Teletraffic Engineering

Oral Exam's Proofs and Notions

August 17, 2018

1. Definition of a Stochastic Process, index space, state space and sample path.
2. Definition of the differences between a discrete-time and continuous-time chain [with plots]
3. [!] Definition of a Markoff Process and Homogeneous Markoff Process.
   (a) Markoff Property in Discrete-Time and Homogeneous Dicrete-Time Markoff Process
   (b) Markoff Property in Continuous-Time and Homogeneous Continuous-Time Markoff Process
4. **Proof** of the PMF (W_i) of time (amount of steps) spent in a state i over discrete-time to get $E\{W_i\}$, the average time spent in a state i.
5. [!] **Proof** of the Chapman-Kolmogoroff Equation in discrete-time to get the CK-Equation in scalar and then matrix form
6. Definition of condition of ergodicity (steady-state) for a discrete-time Markoff chain and ergodic process' matrix.
   (a) Definition of Probability of state occupancy $p_i(n)$
7. **Proof** to find transient behaviour $\underline{p}(n)$ and $\underline{p}_i(n)$ from $\underline{p}(0)$
8. Definition of stationary probability vector $\underline{z}$ and meaning of stationarity of a Markoff chain.
9. Definition of asymptotic or limit probability vector $\underline{p}$
10. **Proof** of the Flow-Conservation Principle from the transient behaviour's equation $\underline{p}(n+1)$ (in scalar form). Meaning and goal of FCP.
    (a) Usage of FCP for transient-behaviour analysis
    (b) Usage of FCP for steady-state analysis [Stationary Equations]
11. Definition of probability of first return to state j in n steps $f_j^{(n)}$ [with plots]
12. Definition of probability of ever returning to state j f_j.
    (a) Classification of a state [Transient vs Recurrent] based on f_j
    (b) Definition of Periodicity of a recurrent state j. Strongly periodic vs weakly periodic state j.
13. Definition of mean recurrence time M_j for recurrent state j.
    (a) Classification of a recurrent state j based on M_j [Positive-Recurrent vs Null-Recurrent]

14. Definition of Irreducible Chain
    (a) 1st Fundamental Theorem for states' classification
    (b) Stationary solution's admittance for an irreducible chain
    (c) Limit solution for ergodic HDTMC
15. 2nd Fundamental Theorem for the ergodicity of a chain. [Infinite-state vs finite-state Markoff chain]
16. Definition of probability of occupancy of a state p_j for a positive recurrent state j.
    (a) M_j and v_{ij}
17. [!] **Proof** of the Chapman-Kolmogoroff Equation in continuous-time to get the CK-Equation in scalar and then matrix form
18. Stationary probability vector and ergodicity condition for a CTMC.
19. [!] Definition of rate transition matrix $\underline{\underline{V}}$ and transition rate v_{ij} for a continuous-time Markoff Chain
    (a) Expression of $h_{ij}(\Delta\tau)$ with the Taylor-MacLaurin Expansion for terms on the main diagonal and outside the main diagonal
    (b) Proof of FCP for a Continuous-time Markoff Chain
20. Sufficient condition for the existence of an ergodic solution of a HCTMC [Finite States vs infinite states]
21. **Proof** of Forward and the Backward CK-Equations in continuous-time case starting from the CK-equation [Relation between $H(t)$ and $\underline{\underline{V}}$].
22. [!] **Proof** of the exponential distribution for the memory-less property of the time spent in a state over continuous time [comparison with discrete-time distribution + plot of exp. distribution for τ, t and $t + \tau$]
23. Definition of Homogeneuous Birth-Death Discrete-Time Markoff Chain [Three-diagonal matrix]
    (a) **Proof** of the Condition of Ergodicity of the chain, applying FCP
    (b) Behaviour of p_i for a Birth - Death DTMC for $b_i = b, d_i = d$
24. Definition of Homogeneuous Birth-Death Continuous-Time Markoff Chain
    (a) FCP for transient analysis
    (b) FCP for stationary analysis
25. [!] **Proof** of a pure Birth HCTMC as a Poisson RV's distribution.
26. Three packet switching architectures and issues related to them, along with solution. Application and usage for them.
27. [!] GEO/GEO/1 queues' parameters analysis for $P\{Service\}$, $P\{Busy\ slot\}$. Model usage and ergodicity condition for it.
28. [!] Solving Chapman-Kolmogoroff Euqation for Pure-Birth HCTMC

(a) **Proof** of the exponential distribution for order-1 Interbirth time [Starting from a pure-birth HCTMC].

29. Definition of Moment generating function $M(s)$.

    (a) Definition of Γ-order Moment

    (b) Definition of Variance of a R.V. $Var\{X\}$

    (c) Definition of Coefficient of Variation of a R.V. C_v

    (d) $E\{X\}, E\{X^2\}, VAR\{X\}$ for the 1-order moment

    (e) $E\{X\}, E\{X^2\}, VAR\{X\}$ for the 2-order moment

30. [!] **Proof** of the PDF of the n-order Interbirth time as an Erlang-n distributed R.V.

    (a) $E\{n\}, VAR\{n\}$

    (b) PDF of the Γ-distribution

31. [!] Discrete-Time Bernoulli Process, Bernoulli Distribution, Binomial Distribution and state Probability $p_i(n)$. Application and usage of Bernoulli Process

    (a) $P_{ON}(t), P_{OFF}(t)$

    (b) State probability $p_n(t)$

    (c) For a Bernoulli Process X: Generating function $G_x(z), E\{X\}, E\{X^2\}, VAR\{X\}$

    (d) For a Bernoulli R.V. Θ: $E\{\Theta\}, E\{\Theta^2\}, VAR\{\Theta\}$

32. [!] Axiomatic definition of a Poisson Process
33. [!] **Proof** of the Poisson Process as limiting case of a discrete-time Bernoulli Process
34. For a Poisson R.V: X: Generating function $G_x(z), E\{X\}, E\{X^2\}, VAR\{X\}$
35. [!] **Proof** that the combination of n independent Poisson processes yields a Poisson process
36. Deterministic Decomposition of a Poisson Process not being a Poisson Process
37. [!] **Proof** of the statistical/probabilistic decomposition of a Poisson Process into n Poisson Processes
38. [!] Continuous-Time Bernoulli Process. State, usage of such process.

    (a) Distribution of Continuous-Time Bernoulli Process

    (b) **Proof** of the Transient Behaviour analysis as binomial distribution of a Continuous-Time Bernoulli Process

39. [!] **Proof** of the PDF of the arrival time over an interval $(0, t)$ as Poisson distribution.
40. Queueing Systems: Kendall's Notation's 6 parameters.
41. Definition of P_B, P_L, P_{BS}, P_D. Definition for Markovian queues.
42. Definition of $E\{T\}, E\{T_S\}, E\{T_W\}, E\{n\}, E\{ns\}, E\{nw\}$ for Markovian queues
43. Definition of Markovian queue' ergodicity condition. State characterization and properties
44. [!] Definition of A, Traffic intensity.

(a) When do we have $E\{n\} = E\{ns\} = A$?

(b) Ergodicity condition of A.

45. [!] **Proof** of the Traversal time $E\{T\}$ in a $M|M|1$ queue, with derivation of $E\{T_W\}$ by the PASTA property.
46. Definition of average values of frequencies $\Lambda_o, \Lambda, \Lambda_L, \Gamma, \Gamma_{Max}$ and their value for a finite Markovian queue
47. [!] $M|M|N_S$ queues, state probability occupancy p_n, p_{NS}, ergodicity condition

    (a) Why do we increase the frequency of termination of service by $n * \mu$ in an $M|M|Ns$ queue?

    (b) **Proof** of $E\{T_W\}$ by the PASTA property in an $M|M|Ns$ queue.

    (c) Ergodicity condition for $M|M|N_s$ queue

    (d) $E\{T_W\}$ for $M|M|N_s$ queue

48. [!] **Proof** of the Erlang-C formula to find P_D in $M|M|Ns$ queues. [Recursive Erlang-C Formula, plot]
49. [!] Performance comparison of $E\{T_S\}$ between:

    (a) $M|M|1$ with one waiting line for one queue

    (b) N_S many $M|M|1$ queues with one waiting line per queue.

    (c) $M|M|N_s$ queue with one waiting line for all N_S servers.

50. $M|M|\infty$ queue. State probability p_n,$E\{T_W\}$ and proof of Poisson Distribution for an $M|M|\infty$ queue
51. [!] **Proof** of the Erlang-B formula to find P_L in $M|M|Ns|0$ queues. Definition and application of the Erlang-B Formula

    (a) Definition of A, Erlang

    (b) Property of the insensibility of the Erlang-B formula

    (c) Recursive form of Erlang-B formula

52. [!] **Proof** of Little's Formula.
53. [!] Definition of Embedded Markoff Chain in an $M|G|1$ queue
54. [!] **Proof** of the Pollaczek-Kinchin Formula in an $M|G|1$ queue ($E\{n\}$, $E\{T_W\}$ in an $M|G|1$ queue) through the mean-value analysis at steady-state
55. Definition of global and local $E\{T_W\}, E\{n\}, E\{ns\}, E\{nw\}$ in an $M|G|1$ queue with no priority classes

    (a) Definition of global and local $E\{T_W\}, E\{n\}, E\{ns\}, E\{nw\}$ with priority classes

56. [!] Definition of virtual and residual time with no priority classes $E\{T_v\}$, $E\{T_R\}$

    (a) Definition of virtual and residual time with priority classes $E\{T_v\}$, $E\{T_R\}$

    (b) **Proof** of $M|G|1$ queue with priorities to find $E\{TW_i\}$

    (c) Conservation law for the virtual time

57. [!] Find p_n in an $M|M|1|N_W$ queue

    (a) **Proof** to find N_ϵ for the percentile of an $M|M|1|N_W$ queue. Meaning and usage of percentiles.

58. [!] **Proof** to find N_ϵ for the percentile in an $M|M|1|\infty$ queues.
59. [!] **Proof** of exponentially-distributed PDF of the waiting time $E\{T_W\}$ in $M|M|1$ queues
60. [!] **Proof** of exponentially distributed PDF of the queueing time $E\{T\}$ in $M|M|1$ queues
61. [!] **Proof** of the Burke Theorem to find that the interdeparture time is independent and exponentially distributed in $M|M|1$ queues. (Markovian nature of a non-markovian queue)
62. Definition of Open Markovian Network of Queues without feedback

    (a) State of the network
    (b) State probability of an open markovian network of queues
    (c) Ergodocity condition of an open markovian network of queues

63. Difference between Open Markovian Network of Queues and Open Network of Markovian Queues
64. Requirements of the Jackson Theorem for Open Markovian Network of queues

    (a) Open Markovian Network of Queues without Feedback vs with Feedback
    (b) [!] "Feeling" of **Proof** of the Jackson Theorem for Open Markovian Network of queues through balance equations

65. Closed Markovian Network of Queues' definition, state probability p_i
66. [!] Gordon-Newell Theorem for a Closed Markovian Network of Queues

    (a) **Proof** of the Gordon-Newell Theorem to come to a product-form solution
    (b) Operating with the Gordon-Newell Theorem [4 steps for this]

67. [!] Average traversal/transit time $E\{T\}$ in a network of queues
68. BCMP Networks' idea and characterization

    (a) State probability definition
    (b) Product-Form solution

TELETRAFFIC ENGINEERING

PROOFS & NOTIONS

1) STOCHASTIC PROCESS:

A stochastic process consists of a family of R.Vs. indexed by a certain index (usually t, time)

$$\{X(t), t \in T\}$$

INDEX SPACE:

It is the set of all possible indexes t of the stochastic process.

STATE SPACE:

It is the set of R.Vs. corresponding to the indexes (of a time series)

EXAMPLE:

Consider the $\boxed{\text{index space}}$ of a time series:

$$\{t_1 \le t_2 \le t_3 \le \dots \le t_n\}$$

$\Rightarrow$ The corresponding $\boxed{\text{state space}}$ is:

$$\{X(t_1) \le X(t_2) \le X(t_3) \le \dots \le X(t_n)\}$$

SAMPLE PATH:

Set of R.Vs. taken at different time indexes of the process.

EXAMPLE:

2) DISCRETE-TIME CHAIN

A discrete-time chain consists of a chain indexed by discrete values t

⇒ DISCRETE INDEX SPACE T

$$t \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

For a DISCRETE-STATE, DISCRETE-TIME CHAIN we hence have:

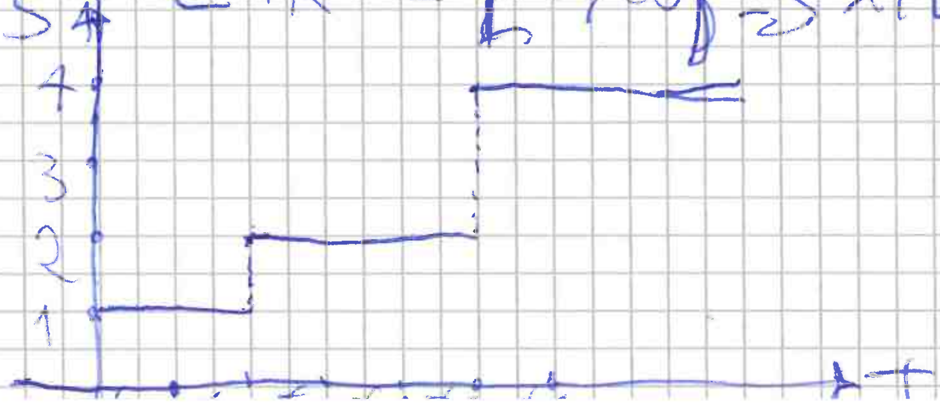
Generally, we take n as a step index over discrete-time:

$$\boxed{X_n \quad n \in \mathbb{N}}$$

CONTINUOUS-TIME CHAIN

A continuous-time chain indexed by values taken over a CONTINUOUS INDEX SPACE T

$$t \in \mathbb{R}^+ = [0, \infty) \Rightarrow x(t), t \in \mathbb{R}^+$$



3) MARKOFF PROCESS:

A stochastic process is said to be a MARKOFF PROCESS if the MARKOVIAN PROPERTY holds for it.

MARKOVIAN PROPERTY:

If the present is given, then the future is CONDITIONALLY INDEPENDENT from the past, and it only depends (conditionally) on the present.

For a TIME SEQUENCE:

$$\boxed{t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n}$$

DISCRETE / CONTINUOUS - STATE SPACE

$$P\{\underbrace{x(t_n) \leq z_n}_{\text{FUTURE}} \mid \underbrace{x(t_{n-1}) = z_{n-1}}_{\text{PRESENT}}, \underbrace{\dots, \cancel{x(t_0) = z_0}}_{\text{PAST}}\}$$

$$= P\{x(t_n) \leq z_n \mid x(t_{n-1}) = z_{n-1}\}$$

HOMOGENEOUS MARKOFF PROCESS:

It is a Markoff process that does not depend on the individual time ~~instants~~ indexes t_n, t_{n-1} but only on the difference between consecutive time indexes

i.e. A shift in time by n-steps is irrelevant $\Rightarrow$ Difference $t - t_n$ only matters

$$\boxed{\begin{aligned} &P\{x(t) \leq z \mid x(t_n) = z_n\} \\ &= P\{x(t - t_n) \leq z \mid x(0) = z_0\} \end{aligned}}$$

CONTINUOUS-STATE / DISCRETE-STATE SPACE

a) DISCRETE-TIME MARKOFF PROPERTY

⇒ Consider the PMF instead of the PDF ($f(x)$) for a DISCRETE-TIME CHAIN.

$$\cancel{P\{X(t_n)=x_n \mid X(t_{n-1})=x_{n-1},\dots,X(t_0)=x_0\} = P\{X(t_n)=x_n \mid X(t_{n-1})=x_{n-1}\}}$$

$$\boxed{P\{X_n = x_n \mid X_{n-1}=x_{n-1},\dots,X_0=x_0\} = P\{X_n = x_n \mid X_{n-1}=x_{n-1}\}}$$

HOMOGENEOUS DISCRETE-TIME MARKOFF PROCESS.

$X_0 = 1$

$$h_{ij}(n) = P\{X_n = j \mid X_{n-1}=i\}$$ ONE-STEP TRANS. PROBABILITY

$$h_{ij}^{(n)} = P\{X_n = j \mid X_0 = i\}$$

$$\Rightarrow \boxed{h_{ij} = P\{X_n = j \mid X_{n-1} = i\}}$$

Only current position matters to next transition

b) CONTINUOUS-TIME MARKOFF PROPERTY

$$\boxed{P\{X(t_n)=x_n \mid X(t_{n-1})=x_{n-1},\dots,X(t_0)=x_0\} = P\{X(t_n)=x_n \mid X(t_{n-1})=x_{n-1}\}}$$

HOMOGENEOUS CONTINUOUS-TIME MARKOFF PROCESS

$$h_{ij}(\tau) = P\{X(t+\tau)=j \mid X(t)=i\}$$

Again only the "current" position in the present matters to determine the next transition

1) PMF & TIME SPENT IN A STATE i

$\boxed{v_i}$ (over DISCRETE-TIME)

⇒ We want to find $\boxed{P\{v_i = n\}}$ (i.e. P. to stay in a state i for n steps)

h_{ii} P. to remain in same state i.

$$P\{v_i = 1\} = 1 - h_{ii}$$

$$P\{v_i = 2\} = h_{ii} \cdot (1 - h_{ii})$$

$$P\{v_i = 3\} = (h_{ii})^2 \cdot (1 - h_{ii})$$

$$\Rightarrow \boxed{P\{v_i = n\} = (h_{ii})^{n-1} \cdot (1 - h_{ii})}$$

⇒ We can now find the $\boxed{E\{v_i\}}$ AVERAGE TIME SPENT in a STATE i

Take the AVG. of the TIME spent (varying 1, 2, 3, ...)

$$E\{v_i\} = \sum_{n=1}^{+\infty} n \cdot (h_{ii})^{n-1} (1 - h_{ii})$$

$$= (1 - h_{ii}) \cdot \sum_{n=1}^{+\infty} n \cdot (h_{ii})^{n-1} = \frac{(1 - h_{ii})}{h_{ii}} \sum_{n=1}^{+\infty} n (h_{ii})^n$$

$$= \frac{\cancel{(1 - h_{ii})}}{\cancel{h_{ii}}} \cdot \frac{\cancel{h_{ii}}}{(1 - h_{ii})^2} = \frac{1}{(1 - h_{ii})}$$

$$\Rightarrow \boxed{E\{v_i\} = \frac{1}{1 - h_{ii}}}$$

⇒ The AVG. time spent in a state (i) is GEOMETRICALLY-DISTRIBUTED

5) PROOF of CK EQUATION in DISCRETE-TIME.

C-K EQUATION in DISCRETE-TIME:

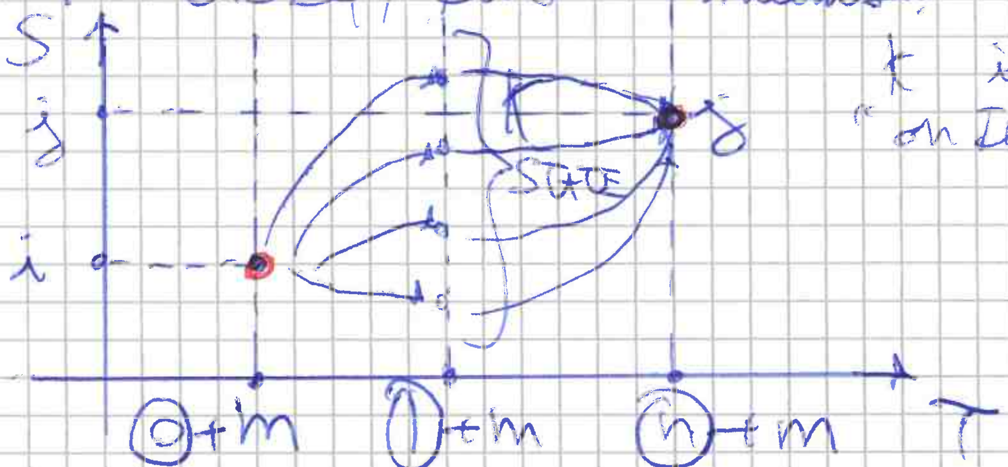
MATRIX FORM

$$\boxed{\underline{\underline{H}}^{(n)} = \underline{\underline{H}}^{(l)} \cdot \underline{\underline{H}}^{(n-l)}}$$

In SCALAR FORM, it is the MATRIX-PRODUCT

$$\boxed{h_{ij}^{(n)} = \sum_{k \in S} h_{ik}^{(1)} \cdot h_{kj}^{(n-1)}}$$

GRAPHICALLY, this means:



k is a state "on the way"

By HOMOGENEITY of a MARKOFF PROCESS:

$$h_{ij}^{(n)} = P\{X_n = j \mid X_0 = i\}$$

$P\{A\} = P\{A|E_1\} + P\{A|E_2\} + \dots$

By the TOTAL PROB. THEOREM:

$$h_{ij}^{(n)} = \sum_{k \in S} P\{X_n = j, X_1 = k \mid X_0 = i\}$$

$P\{A, B\} = P\{A|B\} \cdot P\{B\}$

By the BAYES THEOREM:

$$h_{ij}^{(n)} = \sum_{k \in S} P\{X_n = j \mid X_1 = k, \cancel{X_0 = i}\} \cdot P\{X_1 = k \mid X_0 = i\}$$

$$\Rightarrow \boxed{h_{ij}^{(n)} = \sum_{k \in S} P\{X_n = j \mid X_1 = k\} \cdot P\{X_1 = k \mid X_0 = i\}}$$

$$\boxed{h_{ij}^{(n)} = \sum_{k \in S} h_{kj}^{(n-1)} \cdot h_{ik}^{(1)}}$$

C-K EQUATION $\Rightarrow$ in ~~MATRIX FORM~~ SCALAR FORM

C-K EQUATION in MATRIX FORM:

$$\boxed{\underline{\underline{H}}^{(n)} = \underline{\underline{H}}^{(n-1)} \cdot \underline{\underline{H}}^{(1)}}$$

$\Rightarrow$ COROLLARY:

$$\underline{\underline{H}}^{(n)} = \cancel{\underline{\underline{H}}} \cdot \underline{\underline{H}}^{(n-1)}$$

$$\Rightarrow \underline{\underline{H}}^{(n-1)} = \underline{\underline{H}} \cdot \underline{\underline{H}}^{(n-2)}$$

$$\underline{\underline{H}}^{(n-2)} = \underline{\underline{H}} \cdot \underline{\underline{H}}^{(n-3)}$$

$$\Rightarrow \boxed{\underline{\underline{H}}^{(n)} = \underline{\underline{H}}^{n}}$$ by INDUCTION

6) CONDITION OF ERGODICITY FOR A HOM. DISCRETE-TIME MARKOFF CHAIN.

If the following limit exists, then we have a single solution q [ERGODIC SOLUTION] which is independent from the INITIAL CONDITIONS. (This solution is also stationary)

$$\lim_{n \to \infty} \underline{\underline{H}}^{(n)} = \begin{bmatrix} q \\ q \\ q \\ q \end{bmatrix} = i^T q = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [q]$$

OR:

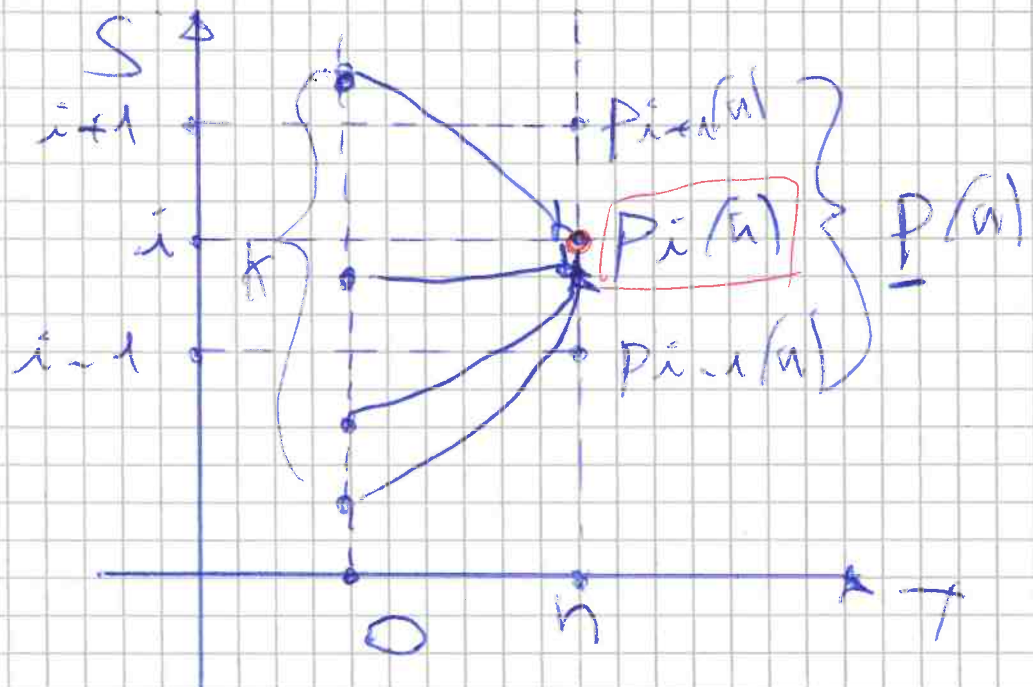
$$\lim_{n\to\infty} \underline{p}(n) = \lim_{n\to\infty} \underline{p}(0)\cdot \underline{\underline{H}}^{(n)} = \lim_{n\to\infty} \underline{p}(0)\cdot \underline{1}\,\underline{q}^{T} = \underline{q}$$

Because $\underline{p}(n) = \underline{p}(0)\cdot \underline{\underline{H}}^{(n)}$

6) PROBABILITY OF STATE OCCUPANCY $\boxed{p_i(n)}$

STATE i

STEP



$$\boxed{p_i(n) = P\{X_n = i\}}$$

= P. to be in state i at step n.

7) TRANSIENT BEHAVIOUR $\boxed{\underline{p}(n)}$, $\boxed{\underline{p}(n+1)}$

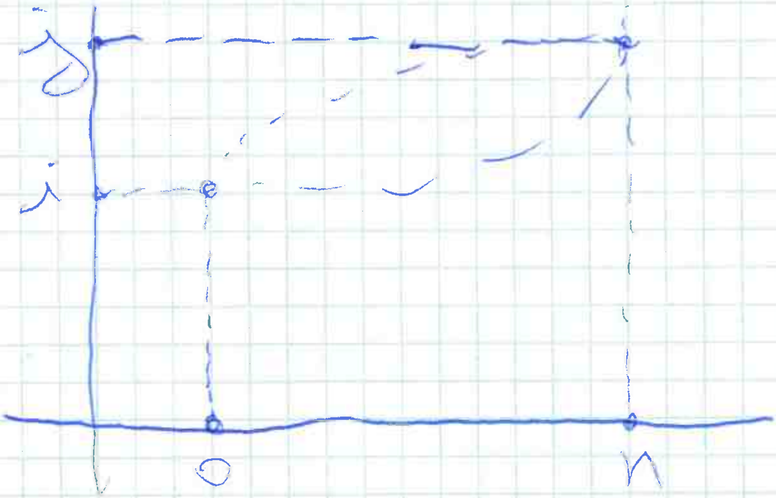
By the TOTAL PROB. THEOREM.

$$p_i(n) = \sum_{k\in S} P\{X_n = i, X_0 = k\}$$

~~$p_i(n)$~~ By the BAYES THEOREM.

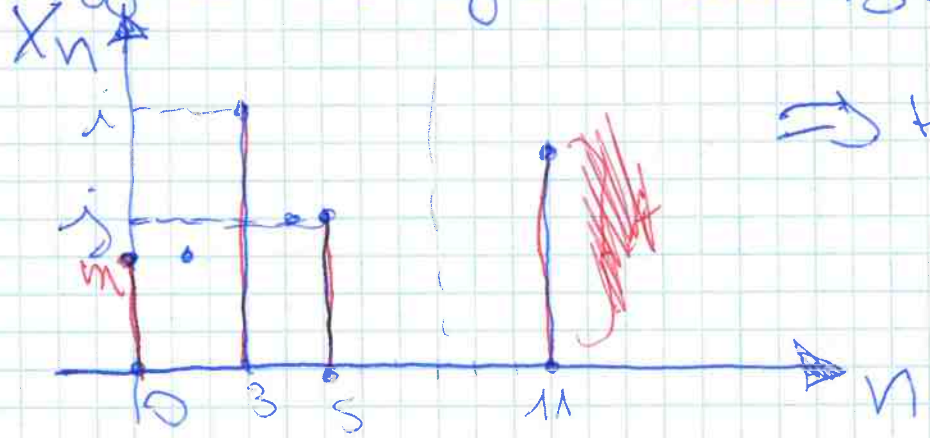
7.5) TRANSITION PROBABILITY in (n) steps:

$$p_{ij}^{(n)} = P\{X_n = j \mid X_0 = i\}$$



JOINT PMF of a MARKOFF CHAIN:

We know the CHARACTERIZATION of a HDT Markoff chain if we know its JOINT PMF.



$\Rightarrow$ HDTMC

$$P\{X_3 = i, X_5 = j, X_{11} = k\}$$

$$= \sum_{m \in S} P\{\overbrace{X_3 = i, X_5 = j, X_{11} = k}^{A}, \overbrace{X_0 = m}^{B}\}$$

$$P\{A, B\} = P\{A \mid B\} \cdot P\{B\}$$

$$P\{X_3 = i \mid X_0 = m\}$$

$$= \boxed{\sum_{m \in S} P\{X_{11} = k, X_5 = j, X_3 = i \mid X_0 = m\} \cdot P\{X_0 = m\}}$$

$$= \sum_{m \in S} P\{X_{11} = k \mid X_5 = j, \cancel{X_3 = i}, \cancel{X_0 = m}\} \cdot P\{X_5 = j \mid X_3

$$= \sum_{m \in S} P\{X_{11}=k \mid X_5=j, \cancel{X_3=i, X_0=m}\} \cdot P\{X_5=j \mid X_3=i\} \cdot P\{X_3=i \mid X_0=m\} \cdot P\{X_0=m\}$$

$$= \sum_{m \in S} h_{jk}^{(6)} \cdot h_{ij}^{(2)} \cdot h_{mi}^{(3)} \cdot p_m(0)$$

ABSORBING STATE / TRAP:

It is a state ~~where~~ that, once ~~...~~ reached, can no longer be escaped from!

↳ It is characterized by the following state PROBABILITY ~~...~~ VECTOR:

$$\underline{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

⇒ When one such state is present in the Markov Chain, then we do NOT consider the chain ERGODIC (although a steady state ~~...~~ exists & a unique asymptotic solution independent from the initial conditions

⇒ exists).

$$p_i(n) = \sum_{k \in S} P\{X_n = i \mid X_0 = k\} \cdot P\{X_0 = k\}$$

$$= \sum_{k \in S} h_{ki}^{(n)} \cdot P\{X_0 = k\}$$

$$= \sum_{k \in S} p_k(0) \cdot h_{ki}^{(n)}$$

$$\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{H}}^n}$$

OR:

$$\boxed{\underline{P}(n+1) = \underline{P}(n) \cdot \underline{\underline{H}}}$$

We can then solve

$$\Rightarrow \begin{cases} \underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{H}}^n \\ \underline{P}(0) = \underline{P}_0 \end{cases}$$

8) STATIONARITY PROBABILITY VECTOR $\underline{Z}$

The stationarity probability vector $\underline{Z}$ is a vector that solves the following equation:

$$\boxed{\underline{Z} = \underline{Z} \cdot \underline{\underline{H}}}$$

(i.e. it is the eigenvector & eigenvalue 1)

$$\boxed{\sum_{i \in S} z_i = 1}$$

$0 \leq z_i \leq 1$. NB: There can be more than one such vector!

[The BEHAVIOUR OF ... among these means ... STATE]

9) LIMIT PROBABILITY VECTOR

$$\boxed{\lim_{n \to \infty} \underline{P}(n) = \underline{P}}$$

$0 \leq p_i \leq 1$

If the limit exists, it is $\sum_i p_i = 1$

(UNIQUE!)

If the LIMIT exists, we can then solve:

$$\boxed{p = p \cdot \underline{\underline{u}}}$$

and p is integrated from the INITIAL CONDITIONS.

(10) FLOW-CONSERVATION PRINCIPLE - PROOF.

$$\underline{p}(n+1) = \underline{p}(n) \cdot \underline{\underline{H}}$$

$\underline{p}(n) = [p_1(n), p_2(n), \ldots p_k(n)]$ → VECTOR containing probabilities of being in every state at step (n).

In $\boxed{\text{SCALAR FORM}}$

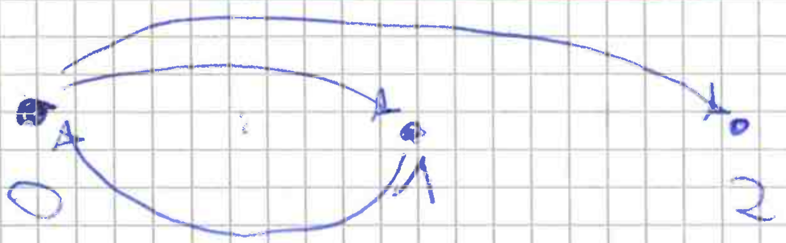
$$p_j(n+1) = \sum_{i \in S} p_i(n) \cdot h_{ij}$$

$$p_j(n+1) = \underbrace{\sum_{i \neq j} p_i(n)\, h_{ij}}_{\text{from } i \text{ to } j} + \underbrace{p_j(n) \cdot h_{jj}}_{\text{from } j \text{ to } j}$$

$$p_j(n+1) = \sum_{i \neq j} p_i(n) \cdot h_{ij} + p_j(n)\left[1 - \sum_{i \neq j} h_{ji}\right]$$

$$\boxed{p_j(n+1) - p_j(n) = \underbrace{\sum_{i \neq j} p_i(n) \cdot h_{ij}}_{\text{ENTERING } j \text{ from } i} - \underbrace{p_j(n) \sum_{i \neq j} h_{ji}}_{\text{LEAVING } j \text{ to } i}}$$

a) Ex. FCP TRANSIENT EQUATIONS



$$p_0(n+1) - p_0(n) = \underbrace{p_1(n) \cdot h_{10}}_{\text{ENTERING } j \text{ from } i} - \underbrace{p_0(n) \cdot [h_{01} + h_{20}]}_{\text{LEAVING } j \text{ to } i}$$

b) FCP STEADY-STATE ⇒ Get rid of n

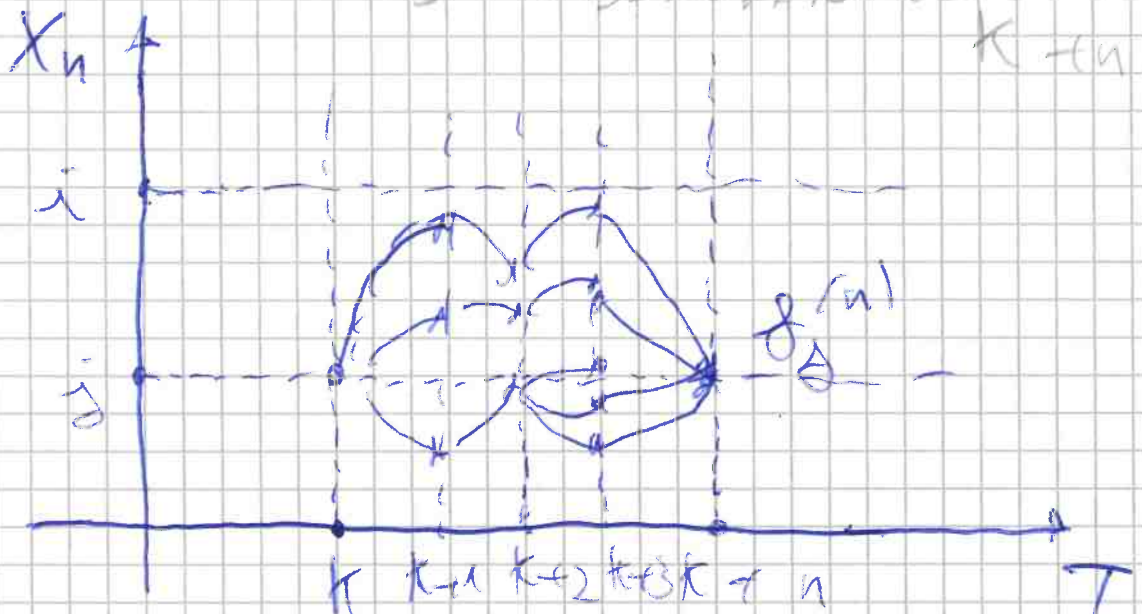
$$0 = p_1 \cancel{(n)} \cdot h_{10} - p_0 \cancel{(n)} [h_{01} + h_{20}]$$

$$\Rightarrow p_1 h_{10} = p_0 [h_{01} + h_{20}]$$

(11) P. of first return in state j in (n) steps: $f_j^{(n)}$

$\boxed{f_j^{(n)}}$ = P{ first return to state j after n steps }

$$f_j^{(n)} = P\{\underbrace{X_{k+1} \neq j, X_{k+2} \neq j, \dots}_{\text{"STEPS IN BETWEEN k-th step and k+n-th step}}, X_{k+n} = j \mid X_k = j\}$$



(12) P. of ever returning to state j: f_j

$$f_j = \sum_{n=1}^{+\infty} f_j^{(n)}$$ ~~Average~~

= Sum of all possible probabilities to go back to state j within an arbitrary amount of steps (n).

(2) f_j

$f_j < 1 \Rightarrow$ TRANSIENT STATE

$f_j = 1 \Rightarrow$ RECURRENT STATE

If $f_j = 1 \Rightarrow$ "We will surely come back to state j"
RECURRENT STATE j

$\Rightarrow \exists n : h_{ij}^{(n)} > 0$

b) If we found that a state is RECURRENT

$\exists n$:

$\Rightarrow \boxed{h_{jj}^{(n)} > 0}$ "There is a finite probability of coming back to state j".

$\Rightarrow$ A state j (RECURRENT) may also be PERIODIC:

$\exists n : \boxed{h_{jj}^{(n)} > 0}$ $n = \#$STEPS

PERIOD $= \boxed{d_j} = $ GCD of $n =$

$d_j > 1$: $\Rightarrow$ STATE j is PERIODIC

$d_j = 1 \Rightarrow$ STATE j is APERIODIC

STRONGLY PERIODIC STATES:

Over multiples of d_j (PERIOD), we always have the same numbers on the main diagonal

WEAKLY PERIODIC STATES:

Over multiples of d_j (PERIOD), we have differing numbers on the main diagonal

(13) MEAN RECURRENCE TIME (M_j) for RECURRENT STATE j

M_j = AVG. amount of time after which we will come back to j.

⇒ Take AVG. for different steps (n)

$$M_j = \sum_{j=1}^{+\infty} n \cdot f_j^{(n)}$$

a)

M_j

(FINITE VALUE)

$\boxed{M_j < \infty}$

POSITIVE RECURRENT STATE j

"We come back to j in finite time"

(INFINITE VALUE)

$\boxed{M_j = \infty}$

NULL-RECURRENT STATE j

"We come back to j in infinite time"

~~(14) 1ST FUNDAMENTAL THEOREM~~

(14) A Markov Chain is said to be IRREDUCIBLE if none of its SUBSETS of STATES is CLOSED.

(i.e. from any state, you can reach any other state)

A subset A of states is said to be CLOSED if it is not possible to move from the states of A to the states of $\bar{A}$.

$$\boxed{\sum_{i \in A} \sum_{j \in \bar{A}} h_{ij} = 0}$$

17) (a) **1ST FUNDAMENTAL THEOREM FOR STATE CLASSIFICATION.**

If a Markov Chain is IRREDUCIBLE, then all of its STATES are of the SAME TYPE.

a) Either all states are TRANSIENT
b) Or all states are NULL-RECURRENT
c) Or all states are POSITIVE-RECURRENT

An IRREDUCIBLE MARKOV CHAIN cannot include TRAP STATES (from it, we cannot reach any other state)

PERIODICITY in MARKOFF CHAINS

In an IRREDUCIBLE MARKOFF CHAIN,

a) Either all states are APERIODIC (PERIOD = 1)
b) Or all states are PERIODIC.

(b) **STATIONARY SOLUTIONS** / PROBABILITIES

A HDTMC can admit multiple asymptotic solutions based on the initial condition, by solving:

$$\boxed{\underline{z} = \underline{z} \cdot \underline{\underline{H}}} \, , \quad \boxed{z_j = \sum_{i \in S} z_j \cdot h_{ji}}$$

(c) **LIMIT SOLUTION for an ERGODIC HDTMC**

A probability distribution p_i, $i \in S$ is said LIMIT PROBABILITY if:

$$\boxed{p_i = \lim_{n \to \infty} p_j(n) \quad \forall i \in S}$$

$$\underline{p} = \lim_{n \to \infty} \underline{p}(n)$$

(15) 2nd FUNDAMENTAL THEOREM:

For all HDTMC, APERIODIC, IRREDUCIBLE:

If $\boxed{\lim_{n \to \infty} p_j(n)}$ EXISTS and is independent from the INITIAL CONDITIONS.

[⇒ A steady-state exists]

IF all states are TRANSIENT or NULL-RECURRENT

$\boxed{p_j = 0}$ $\forall j \in ES$

LIMIT PROBABILITY

IF all states are POSITIVE RECURRENT

$\boxed{p_j > 0}$ $\forall j \in ES$

⇓

LIMIT PROBABILITIES are also a STATIONARY DISTRIBUTION

⇓

All states $\in ES$ are ERGODIC & the chain is ERGODIC.

NB: For a HDTMC, APERIODIC, IRREDUCIBLE, all POSITIVE-RECURRENT STATES ⇒ All states are ERGODIC.

⇓

The chain is ERGODIC.

a) FINITE # STATES IRREDUCIBLE, APERIODIC MARKOFF CHAIN

⇓

ERGODIC CHAIN

INFINITE # STATES IRREDUCIBLE, APERIODIC MARKOFF CHAIN

If $\boxed{p_j = 0}$ $\forall$ ⇒ All states TRANSIENT or NULL-RECURRENT

If $p_j >$ All states POSITIVE RECURRENT

INFINITE # STATES, APERIODIC, IRREDUCIBLE MARKOFF CHAIN

$P_j = 0$ All NULL-RECURRENT STATES

$P_j > 0$ All POSITIVE-RECURRENT STATES ⇒ ERGODIC CHAIN

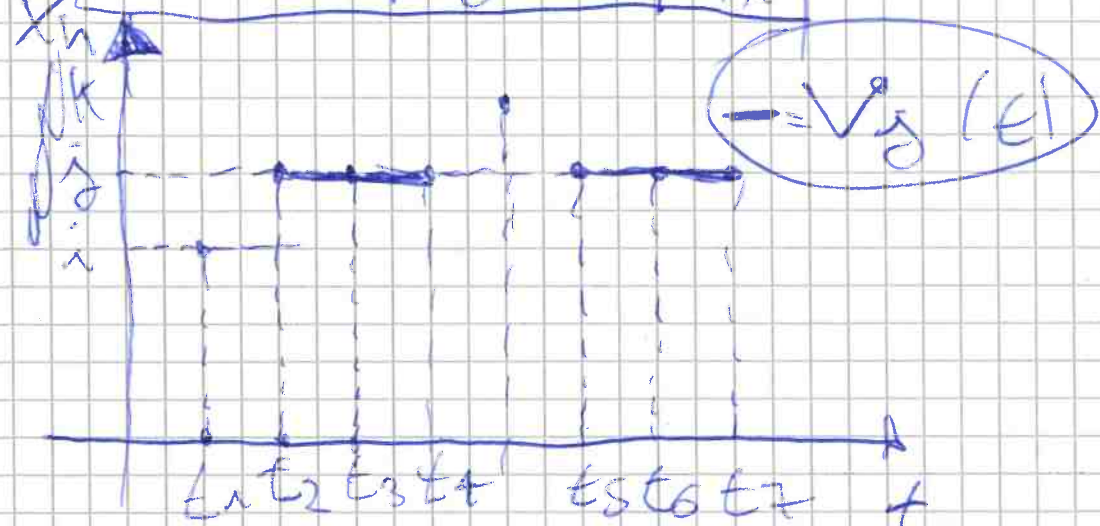
(16) For a POSITIVE-RECURRENT STATE j, AT STEADY-STATE → P_j

$$P_j = \lim_{T \to \infty} \frac{V_j(T)}{T}$$

$V_j(t)$ = time spent in state j

$\boxed{V_{ij}}$ = AVG. # VISITS to a STATE i between two successive visits to state j.

$$\boxed{V_{ij} = \frac{P_i}{P_j} = \frac{M_j}{M_i}}$$

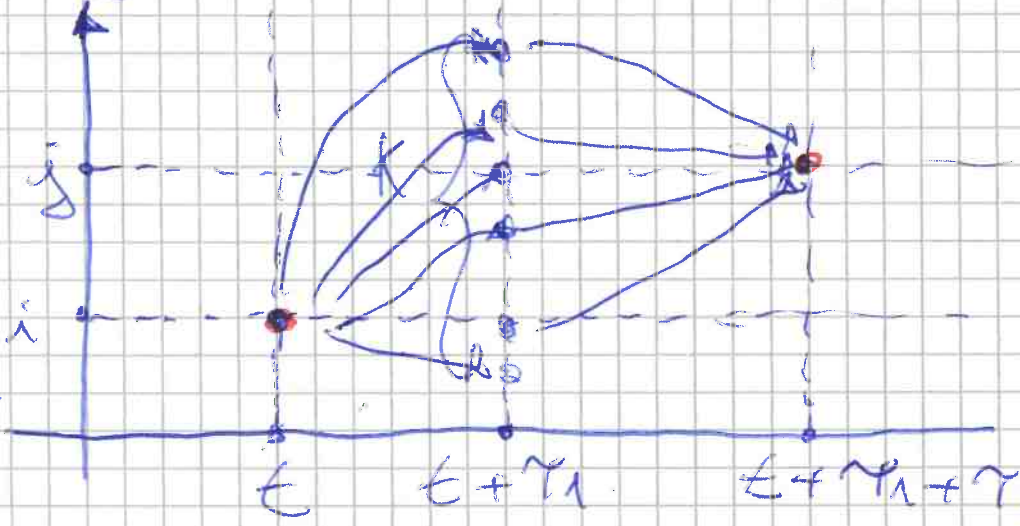


$$\boxed{M_j = \frac{1}{P_j}} \qquad \boxed{P_j = \frac{1}{M_j}}$$

17) CK-EQUATION in CONTINUOUS-TIME

$$\boxed{\underline{H}(\tau_1+\tau_2)=\underline{H}(\tau_1)\cdot\underline{H}(\tau_2)}$$

IN GRAPHICAL FORM, we have:



PROOF, in scalar form:

$$h_{ij}(\tau_1+\tau_2)=P\{X(t+\tau_1+\tau_2)=j \mid X(t)=i\}$$

By the TOTAL PROBABILITY THEOREM:

$$h_{ij}(\tau_1+\tau_2)=\sum_{k\in S}P\{X(t+\tau_1+\tau_2)=j,\ X(t+\tau_1)=k \mid X(t)=i\}$$

By the BAYES THEOREM: $P\{A,B\}=P\{A|B\}\,P\{B\}$

$$=\sum_{k\in S}P\{X(t+\tau_1+\tau_2)=j \mid X(t+\tau_1)=k,\ \cancel{X(t)=i}\}\cdot P\{X(t+\tau_1)=k \mid X(t)=i\}$$

$$=\sum_{k\in S}P\{X(t+\tau_1+\tau_2)=j \mid X(t+\tau_1)=k\}\,P\{X(t+\tau_1)=k \mid X(t)=i\}$$

$$=\sum_{k\in S}h_{kj}(\tau_2)\cdot h_{ik}(\tau_1)$$

IN MATRIX FORM:

$$\boxed{\underline{H}(\tau_1+\tau_2)=\underline{H}(\tau_2)\cdot\underline{H}(\tau_1)}$$

18 STATIONARITY PROBABILITY VECTOR IN CONTINUOUS-TIME:

$X(t), t \geq 0$

$$\boxed{\underline{P} = \underline{P} \cdot \underline{\underline{H}}(\tau)} \quad \forall \tau$$

(NB: There may be multiple STATIONARY PROBABILITY VECTORS)

ERGODICITY CONDITION - CONTINUOUS-TIME

If:

$$\boxed{\lim_{t \to \infty} \underline{P}(t) = \underline{P}}$$

Then the limit exists & it is independent from the INITIAL CONDITIONS & it is also the STATIONARY PROBABILITY VECTOR

19 RATE TRANSITION MATRIX $\underline{V}$.

$\underline{V} = [v_{ij}]$ It is the cell-wise derivative for $\tau = 0$.

RATE TRANSITION v_{ij}:

$$\boxed{v_{ij} = \frac{d}{d\tau} h_{ij}(\tau) \Big|_{\tau=0}}$$

[MATRIX of NUMBERS, not functions]

NORMALIZATION CONDITION for $\underline{V}$:

$$\boxed{\sum_{j \in S} v_{ij} = 0}$$

(A row sums up to 0)

NORMALIZATION CONDITION for $\underline{\underline{H}}(\tau)$

$$\boxed{\sum_{j \in S} h_{ij}(\tau) = 1}$$

(A) $\boxed{h_{ij}(\Delta\tau)}$ with TAYLOR-MACLAURIN EXPANSION.

We know that:

$$v_{ij} = \frac{d}{d\tau} h_{ij}(\tau) \Big|_{\tau=0}$$

For a very small interval $\boxed{\Delta\tau}$, take the TAYLOR-MACLAURIN EXPANSION of $h_{ij}(\Delta\tau)$

$$\boxed{\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n}$$

⇒ For $h_{ij}(\Delta\tau)$:

INFINITESIMAL OF HIGHER ORDER

$$\boxed{h_{ij}(\Delta\tau) = h_{ij}(0) + v_{ij} \cdot \Delta\tau + o(\Delta\tau)}$$

⇒ This is a LINEAR APPROXIMATION of $h_{ij}(\Delta\tau)$ [but first 2 terms are enough]

⇒ Through $h_{ij}(\Delta\tau)$, we can express components on the main diagonal as the subtraction of the ~~...~~ ones NOT on the main diagonal.

$$v_{ii} = -\sum_{j \neq i} v_{ij}$$

For $\underline{\underline{V}}$:

12 5 TRANSIENT BEHAVIOUR for. a MCTMC.

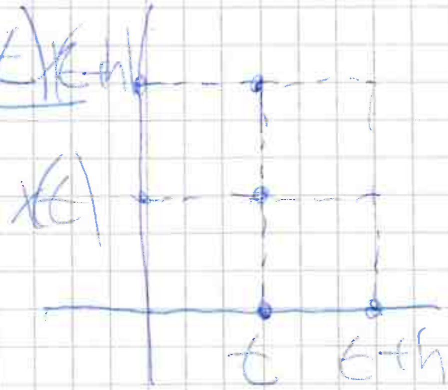
We know: $\left\{\begin{array}{l}\boxed{\underline{P}(t+\tau) = \underline{P}(t)\cdot\underline{\underline{H}}(\tau)} \\ \underline{\underline{H}}(0) = \underline{\underline{I}}\end{array}\right.$

$$\underline{P}(t+\tau) - \underline{P}(t) = \underline{P}(t)\cdot\underline{\underline{H}}(\tau) - \underline{P}(t)$$

$$\lim_{\tau\to 0}\frac{\underline{P}(t+\tau)-\underline{P}(t)}{\tau} = \lim_{\tau\to 0}\underline{P}(t)\cdot\left[\frac{\underline{\underline{H}}(\tau)-\underline{\underline{H}}(0)}{\tau}\right]$$

We know the definition of DERIVATIVE

$$\frac{d}{dt}x(t) = \lim_{h\to 0}\frac{x(t+h)-x(t)}{h}$$



$$\Rightarrow \boxed{\left\{\begin{array}{l}\frac{d}{dt}\underline{P}(t) = \underline{P}(t)\cdot\underline{\underline{V}} \\ \underline{P}(0) = \underline{P}_0\end{array}\right.}$$

$\rightarrow \underline{\underline{V}} = \frac{d}{d\tau}h_{ij}\Big|_{\tau=0}$

$$\Rightarrow \frac{d}{dt}p_j(t) = \sum_{i\in S} p_i(t)\, v_{ij}$$

IN SCALAR FORM!

$$\underline{P}(t) = \underline{P}(0)\, e^{\underline{\underline{V}}t} = \underline{P}(0)\left[\underline{\underline{I}} + \sum_{n=1}^{+\infty}\frac{\underline{\underline{V}}^n t^n}{n!}\right]$$

ERGODIC BEHAVIOUR (i.e. if the CHAIN is ERGODIC) ⇒ A steady-state EXISTS

$$\begin{cases} \underline{P} \cdot \underline{\underline{V}} = 0 \\ \underline{P} \cdot \underline{1}^T = 1 \end{cases}$$

$$\underline{V} = \begin{bmatrix} \overset{j=i}{\ddots} \end{bmatrix}$$

[balances the ones outside i]

$$v_{ij} = \begin{cases} v_{ij} \leq 0 & \text{if } j = i \quad [\text{MAIN DIAGONAL} \Rightarrow \text{NEG}] \\ v_{ij} \geq 0 & \text{if } j \neq i \quad [\text{OUTSIDE MAIN DIAGONAL IS POSITIVE}] \end{cases}$$

Also:

$$h_{ij}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\Rightarrow$ We can then write $h_{ij}(\Delta\tau)$ for the two cases, based on the value of $h_{ij}(0)$

$$h_{ij}(\Delta\tau) = h_{ij}(0) + v_{ij}\,\Delta\tau + o(\Delta\tau)$$

(outside main diagonal)

$$\boxed{\Rightarrow h_{ij}(\Delta\tau) = \begin{cases} v_{ij}\Delta\tau + o(\Delta\tau) & \text{if } j \neq i \\ 1 - \sum_{i \neq j} v_{ij}\Delta\tau + o(\Delta\tau) & \text{if } j = i \end{cases}}$$

(opposite of main diagonal's value)

LINEAR BEHAVIOUR

PROOF of FCP in CONTINUOUS-TIME

$$v_{jj} = -\sum_{i \neq j} v_{ij} \qquad \boxed{\frac{d}{dt} p_j(t) = \sum_{i \in S} p_i(t)\, v_{ij}}$$

$$\frac{d}{dt} p_j(t) = \sum_{i \neq j} p_i(t) \cdot v_{ij} + p_j(t) \cdot v_{jj}$$

[ONLY ... STAY IN SAME STATE]

$$\Rightarrow \boxed{\frac{d}{dt} p_j(t) = \underbrace{\sum_{i \neq j} p_i(t) \cdot p_{ij}(t)}_{\text{ENTERING } j \text{ from } i} - \underbrace{p_j(t) \sum_{i \neq j} \nu_{ji}}_{\text{LEAVING } j \text{ to } i}}$$

(20) **SUFFICIENT CONDITION FOR THE EXISTENCE OF ERGODIC SOLUTION FOR HCTMC:**

- FINITE # STATES

An IRREDUCIBLE HCTMC is ERGODIC

- INFINITE # STATES, IRREDUCIBLE HCTMC

All NULL-RECURRENT STATES:

$$\exists \lim_{t \to \infty} p_j(t) = 0$$

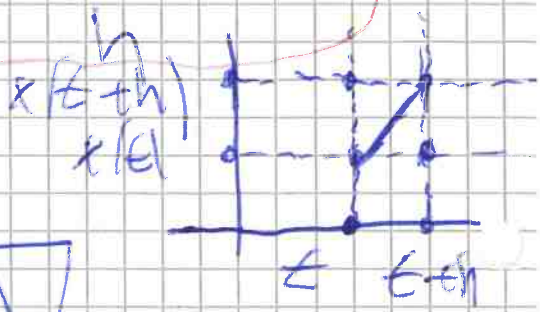
All POSITIVE-RECURRENT STATES:

$$\exists \lim_{t \to \infty} p_j(t) = p_j > 0$$

(21) **PROOF OF FORWARD & BACKWARD CHAPMAN-KOLMOGOROFF EQUATION** (CONTINUOUS TIME)

Recall the definition of DERIVATIVE:

$$\frac{d}{dt} x(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$



Recall the C-K CONTINUOUS-TIME EQUATION:

$$\boxed{\underline{H}(t+\theta) = \underline{H}(t) \cdot \underline{H}(\theta)}$$

For $\theta > 0$:

$$\boxed{\underline{\underline{H}}(t+\theta) = \underline{\underline{H}}(t) \cdot \underline{\underline{H}}(\theta)}$$

$$\boxed{\underline{\underline{H}}(0) = \underline{\underline{I}}}$$

NEW

$$\underline{\underline{H}}(t+\theta) - \underline{\underline{H}}(t) = \underline{\underline{H}}(t) \cdot \underline{\underline{H}}(\theta) - \underbrace{\underline{\underline{H}}(t)}_{\underline{\underline{H}}(t)\,\underline{\underline{I}}} \underline{\underline{I}} = \underline{\underline{H}}(0)$$

$$\underbrace{\lim_{\theta \to 0^+} \frac{\underline{\underline{H}}(t+\theta) - \underline{\underline{H}}(t)}{\theta}} = \lim_{\theta \to 0^+} \underline{\underline{H}}(t) \underbrace{\frac{\underline{\underline{H}}(\theta) - \underline{\underline{H}}(0)}{\theta}}_{\left.\frac{d}{dt}\underline{\underline{H}}(t)\right|_{t=0} = \underline{\underline{V}}}$$

$$\begin{cases} \frac{d}{dt}\underline{\underline{H}}(t) = \underline{\underline{H}}(t) \cdot \underline{\underline{V}} \\ \underline{\underline{H}}(0) = \underline{\underline{I}} \end{cases}$$

FORWARD C-K EQUATION

$$\begin{cases} \frac{d}{dt}\underline{\underline{H}}(t) = \underline{\underline{V}} \cdot \underline{\underline{H}}(t) \\ \underline{\underline{H}}(0) = \underline{\underline{I}} \end{cases}$$

BACKWARD CK EQUATION

21 PROOF of the EXPONENTIAL DISTRIBUTION for the MEMORYLESS PROPERTY of ~~[illegible]~~ TIME SPENT in A STATE W [CONTINUOUS TIME]

We know the exponential distribution has PDF for X:

~~$f_X(x) = \lambda \cdot e^{-\lambda x} \cdot \mu(x)$~~

GENERAL: Seen before!

$$\boxed{f_W(x) = \lambda \cdot e^{-\lambda x} \cdot \mu(x)}$$

PDF of EXPONENTIAL DISTRIBUTION for time spent in a state $\Rightarrow W$.

~~PROOF~~: MEMORYLESS PROPERTY:

EX: t = ONE HOUR; $\tau = \frac{1}{2}$ HOUR

$$\boxed{P\{W > t + \tau \mid W > t\} = P\{W > \tau\}}$$

"The probability that my phone call will last longer than one hour and a half, KNOWING it has already lasted one hour = P. that my phone call will last longer than half an hour"

[i.e. The past is totally irrelevant for the future. Only the present matters to find it]

~~COMPLEMENTING [illegible]~~

PROOF: Complementing & substituting t by τ:

"⇒ PHONE CALL lasting less than" X "

$$P\{W \le t + \tau \mid W > \tau\} = P\{W \le t\}$$

By the BAYES THEOREM:

$$P\{A \mid B\} = \frac{P\{A, B\}}{P\{B\}}$$

$$\frac{P\{w \le t+\tau,\ w > \tau\}}{P\{w > \tau\}} = P\{w \le t\}.$$

$$= \frac{P\{\tau < w \le t+\tau\}}{P\{w > \tau\}} = P\{w \le t\}$$

We know the ~~PROBABILITY DISTRIBUTION~~ FUNCTION $F_w(x)$

$$\boxed{F_w(x) = (1 - e^{-\lambda x})\,\mu(x) = P\{w \le x\}}$$

$$= \frac{F_w(t+\tau) - F_w(\tau)}{1 - F_w(\tau)} = F_w(t) - \underbrace{F_w(0)}_{0}$$

$$= F_w(t+\tau) - F_w(\tau) = [F_w(t) - F_w(0)][1 - F_w(\tau)]$$

Take the LIMIT for $t \to 0$.

$$\lim_{t\to 0} \frac{F_w(t+\tau) - F_w(\tau)}{t} = \lim_{t\to 0} \frac{F_w(t) - F_w(0)}{t}[1 - F_w(\tau)]$$

$$F_w'(\tau) = F_w'(0)[1 - F_w(\tau)]$$

Now set $\boxed{A(\tau) = 1 - F_w(\tau)} = e^{-\lambda t}\mu(t)$

$$\Rightarrow \boxed{F_w(\tau) = 1 - A(\tau)} = 1 - e^{-\lambda t}\mu(t)$$

$$\Rightarrow \boxed{F_w'(\tau) = -A'(\tau)} = \lambda \cdot e^{-\lambda t}$$

$$A'(\tau) = -\lambda \cdot e^{-\lambda t}\mu(t)$$

We know: $F_w(\tau) = \left[1 - e^{-\lambda t}\right] \mu(t)$

$\Rightarrow$ For $\tau = 0$:

$F_w'(0) = \lambda \qquad F_w(0) = 0$

$A(0) = 1 \qquad A'(0) = \lambda$

$$\Rightarrow \underbrace{F_w'(\tau)}_{-A'(\tau)} = \underbrace{F_w'(0)}_{\lambda} \underbrace{\left[1 - F_w(\tau)\right]}_{A(\tau)}$$

$$\Rightarrow \begin{cases} -A'(\tau) = \lambda \cdot A(\tau) \\ A(0) = 1 \end{cases} \qquad \begin{cases} A'(\tau) = -\lambda A(\tau) \\ A(0) = 1 \end{cases}$$

DIFFERENTIAL EQUATION, solvable via $\mathcal{L}$-Transform.

$\Rightarrow \boxed{\mathcal{L}} \Rightarrow s \cdot A(s) - A(0) = -\lambda A(s)$

$A(s) \cdot (s + \lambda) = 1$

$$\Rightarrow \boxed{A(s) = \frac{1}{s+\lambda}}$$

$$\Rightarrow \mathcal{L}^{-1} \Rightarrow \boxed{A(\tau) = e^{-\lambda\tau} \mu(\tau)} ~~= 1 - F_w(\tau)~~$$

$F_w(\tau) = 1 - A(\tau) = \left[1 - e^{-\lambda\tau}\right] \mu(\tau)$

$\hookrightarrow$ EXPONENTIAL PROB. DISTRIBUTION FUNCTION

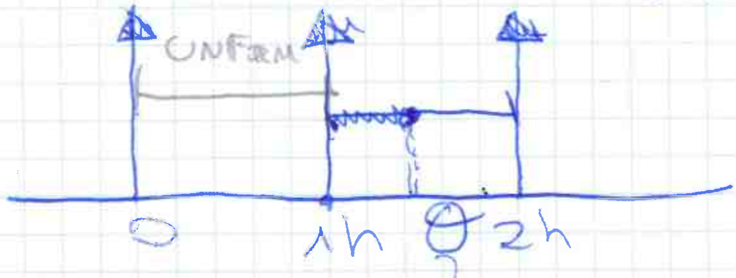
$$\boxed{f_w(\tau) = \lambda \cdot e^{-\lambda\tau} \cdot \mu(\tau)}$$

$\hookrightarrow$ EXPONENTIAL PDF of w, TIME SPENT IN A STATE

(AFTER PLOT)

AVERAGE RESIDUAL TIME'S PARADOX:

- If we have DETERMINISTIC (uniform) distribution of the arrival times.



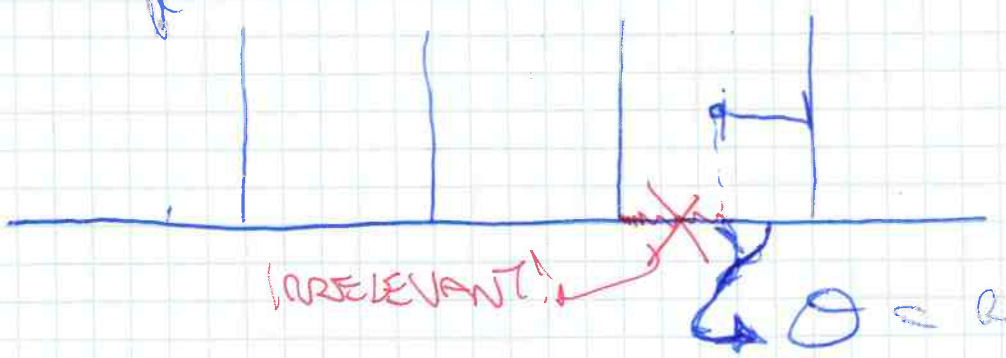
$\lambda = 1h$

If arriving to the BUS STATION randomly. $\Rightarrow$ $E\{\theta\} = 0.5$

θ = RESIDUAL TIME.

NB: The residual time is the time remaining to be waited for the next arrival to occur. (i.e: next bus' arrival).

- If we have arrivals from a POISSON PROCESS. (i.e: arrivals are exponentially distributed). with same AVG. arrival rate.



θ = RESIDUAL TIME.

PARADOX of the RESIDUAL TIME!

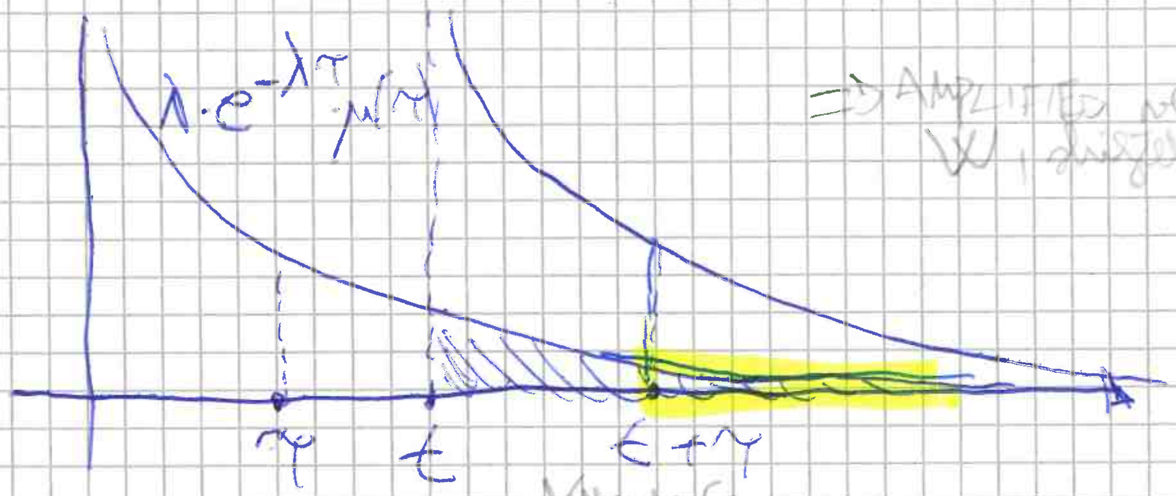
$E\{\theta\} = 1$ (because MEMORYLESS

Forget about the PAST!

↳ Useless!

(because arrivals are exponentially distributed) & the MEMORYLESS property holds.

So, in CONTINUOUS-TIME, we have:



$\Rightarrow$ AMPLIFIED version of $\mathcal{U}$, shifted in time.

$$P\{\mathcal{U} > t+\tau \mid \mathcal{U} > t\} = P\{\mathcal{U} > \tau\}$$

$\Rightarrow$ MEMORYLESS PROPERTY HOLDS

Even if you shift over time, you still get an ~~equal~~ EXPONENTIALLY-DISTRIBUTED copy of the distribution of TIME spent in a state (PDF) $\mathcal{U}$.

(This only happens if you have exponential distribution)

(23) HOMOGENEOUS BIRTH-DEATH DISCRETE-TIME MARKOFF CHAIN

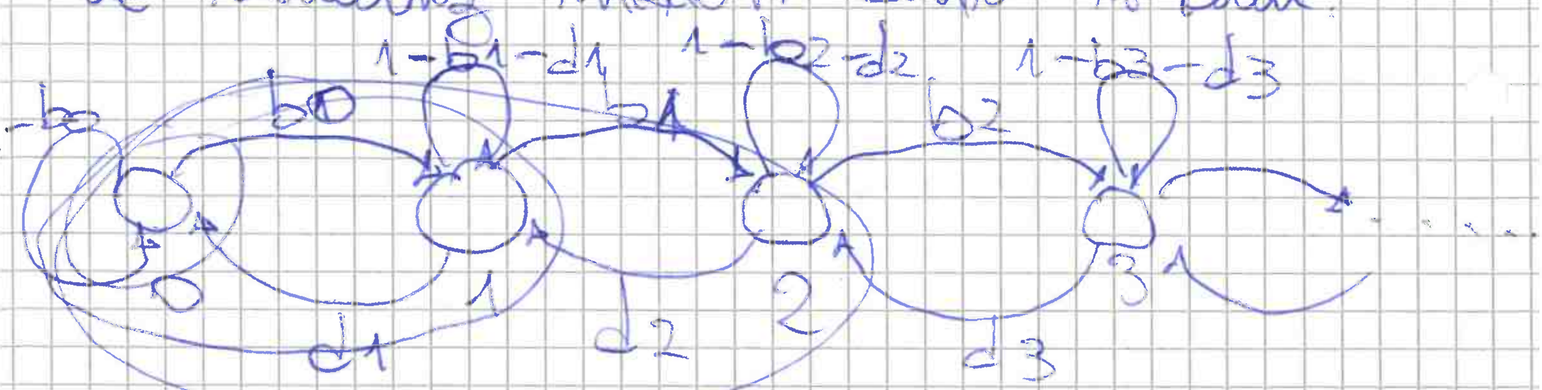
A HOMOGENEOUS BIRTH-DEATH DT Markoff Chain is a Markoff Chain where only 4 types of transitions are possible:

- BIRTH b_i $j=i+1$
- DEATH d_i $j=i-1$
- 1 BIRTH b_i & 1 DEATH d_i $j=i$
- NO BIRTH, NO DEATH $j=i=0$

$\Rightarrow$ This results in the following TRANSITION

$$h_{ij} = \begin{cases} b_i & j=i+1,\ i\geq 0 \\ d_i & j=i-1,\ i\geq 1 \\ 1-b_i-d_i & j=i \\ 1-b_0 & j=i=0 \end{cases}$$

The resulting MARKOFF CHAIN is then.



And the TRANSITION MATRIX $\underline{\underline{H}}$ is:

$$\underline{\underline{H}} = \begin{array}{c|cccccc} & 0 & 1 & 2 & & & \\ 0 & 1-b_0 & b_0 & 0 & 0 & & \\ 1 & d_1 & 1-b_1-d_1 & b_1 & 0 & & \\ 2 & 0 & d_2 & 1-b_2-d_2 & b_2 & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ & & & & d_i & 1-b_i-d_i & b_i \end{array}$$

$\boxed{\text{TRI-DIAGONAL TRANSITION MATRIX}}$

(a) ERGODICITY CONDITION:

Infinite # STATES, APERIODIC?

$0 < b_i < 1$

$0 < d_i < 1$

$\boxed{\text{APERIODIC}}$

$b_0 < 1 \Rightarrow$ RIPPLE

$b_i + d_i \neq 1$

$b_0 = 1$

$\Rightarrow (b_i + d_i = 1)$

$\Rightarrow$ PERIODIC with PERIOD = 2

[NB: NO RIPPLES]

+ Are all states POSITIVE RECURRENT?

(i.e. $\boxed{p_i > 0\ ?}$ $\forall i \in S$)

⇒ Apply FCP to the GRAPH:

$$d_1 \cdot p_1 = b_0 \cdot p_0 \Rightarrow p_1 = \frac{b_0 \cdot p_0}{d_1} = \frac{b_0}{d_1} \cdot p_0$$

$$d_2 \cdot p_2 = b_1 \cdot p_1 \Rightarrow p_2 = \frac{b_1 \cdot p_1}{d_2} = \frac{b_1}{d_2} \frac{b_0}{d_1} p_0$$

$$d_3 \cdot p_3 = b_2 \cdot p_2 \Rightarrow p_3 = \frac{b_2 \cdot p_2}{d_3} = \frac{b_2}{d_3} \frac{b_1}{d_2} \frac{b_0}{d_1}$$

In general:

$$\Rightarrow \cancel{\boxed{p_{i+1} = \frac{b_{i-1}\, p_{i+1}}{d_i}}} \quad \boxed{p_i = \frac{b_{i-1} \cdot p_{i-1}}{d_i}}$$

AND:

$$\boxed{p_i = \frac{b_0 \cdot b_1 \cdot b_2 \cdots b_{i-1}}{d_1 \cdot d_2 \cdot d_3 \cdots d_i} \cdot p_0}$$

$$\Rightarrow \boxed{p_i = p_0 \cdot \prod_{j=0}^{i-1} \frac{b_j}{d_{j+1}}}$$

⇒ We can now apply the NORMALIZATION CONDITION

$$\boxed{\sum_{i=0}^{+\infty} p_i = 1} \Rightarrow \sum_{i=0}^{+\infty} p_0 \cdot \prod_{j=0}^{i-1} \frac{b_j}{d_{j+1}} = 1$$

Take p_0 out:

$$p_0 + \sum_{i=1}^{+\infty} p_0 \cdot \prod_{j=0}^{i-1} \frac{b_j}{d_{j+1}} = 1$$

$$p_0 \left[1 + \sum_{i=1}^{+\infty} \prod_{j=0}^{i-1} \frac{b_j}{d_{j+1}} \right] = 1$$

$$\Rightarrow p_0 = \frac{1}{1 + \sum_{i=1}^{+\infty} \prod_{j=0}^{i-1} \frac{b_j}{d_{j+1}}}$$

If the series DIVERGES
$\Downarrow$
All NULL RECURRENT STATES

$p_0 = 0$
$\Rightarrow p_i = 0$

"We'll never come back to them",
MORE BIRTHS THAN DEATHS: $b_i \geq d_{j+1}$

If the series CONVERGES
$\Downarrow$
All POSITIVE RECURRENT STATES $\Rightarrow$ $\boxed{\text{ERGODIC CHAIN}}$

$0 < p_0 < 1$
$\Rightarrow 0 < p_i < 1$

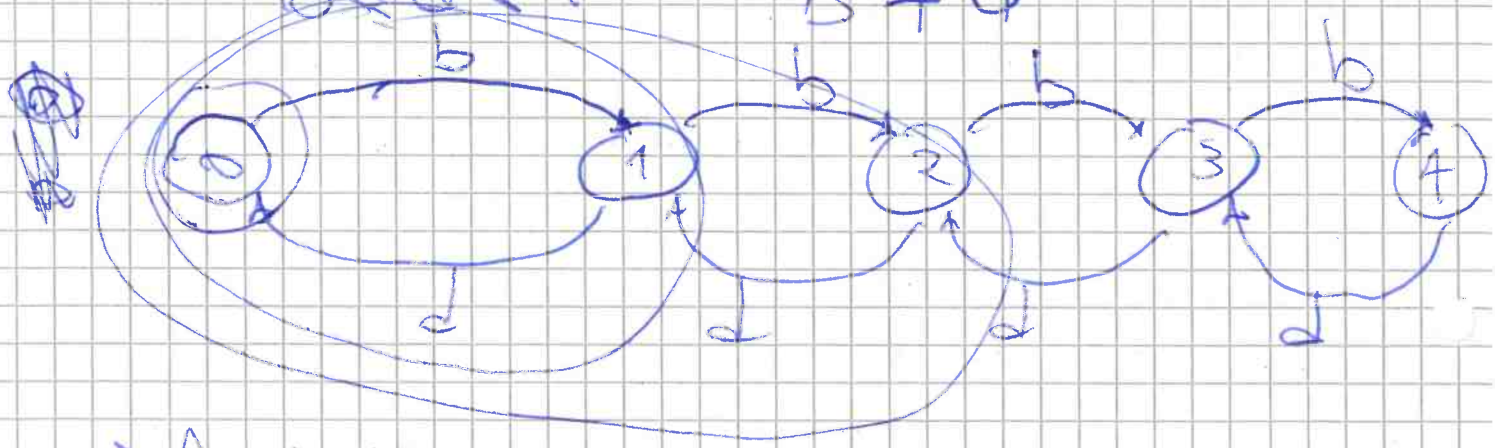
This occurs if:

$\exists j \geq \bar{j} \; \frac{b_j}{d_{j+1}} < 1$

$\Rightarrow \boxed{b_j < d_{j+1}}$

(b) Assume $b_i = b$, $d_i = d$ (APERIODIC CHAIN)

$\Rightarrow 0 < b < 1$ $\quad b + d \neq 1$
$0 < d < 1$ $\quad b \neq 0$



$\Rightarrow$ Apply FCP.

$$\begin{cases} b \cdot p_0 = d \cdot p_1 \Rightarrow p_1 = \frac{b}{d} p_0 \\ b \cdot p_1 = d \cdot p_2 \\ b \cdot p_2 = d \cdot p_3 \quad \Rightarrow p_2 = \frac{b}{d} p_1 = \frac{b}{d}\frac{b}{d} p_0 \\ b \cdot p_3 = d \cdot p_4 \end{cases}$$

$$\Rightarrow p_3 = \frac{b}{d} p_2 = \frac{b}{d}\frac{b}{d}\frac{b}{d} p_0$$

$$\Rightarrow \boxed{p_i = \left(\frac{b}{d}\right)^i p_0}$$

$\Rightarrow$ Apply the NORMALIZATION CONDITION:

$$\sum_{i=0}^{+\infty} p_i = 1 \Rightarrow \sum_{i=0}^{+\infty} \left(\frac{b}{d}\right)^i p_0 = 1 \Rightarrow p_0 \sum_{i=0}^{+\infty} \left(\frac{b}{d}\right)^i = 1$$

$$\Rightarrow \boxed{p_0 = \frac{1}{\sum_{i=0}^{+\infty} \left(\frac{b}{d}\right)^i}}$$

$$\sum_{i=0}^{+\infty} (\alpha)^i = \frac{1}{\alpha}$$

If $\frac{b}{d} \geq 1$

$\Rightarrow$ SERIES DIVERGES

$\Downarrow$

All states ~~[illegible]~~ NULL RECURRENT ~~REMAIN NOT ERGODIC~~

$\Rightarrow p_0 = 0$

$\Rightarrow p_i = 0$

If $\frac{b}{d} < 1$

$\Rightarrow$ Series CONVERGES

$\Downarrow$

All states POSITIVE RECURRENT

$\Downarrow$

Chain is ERGODIC

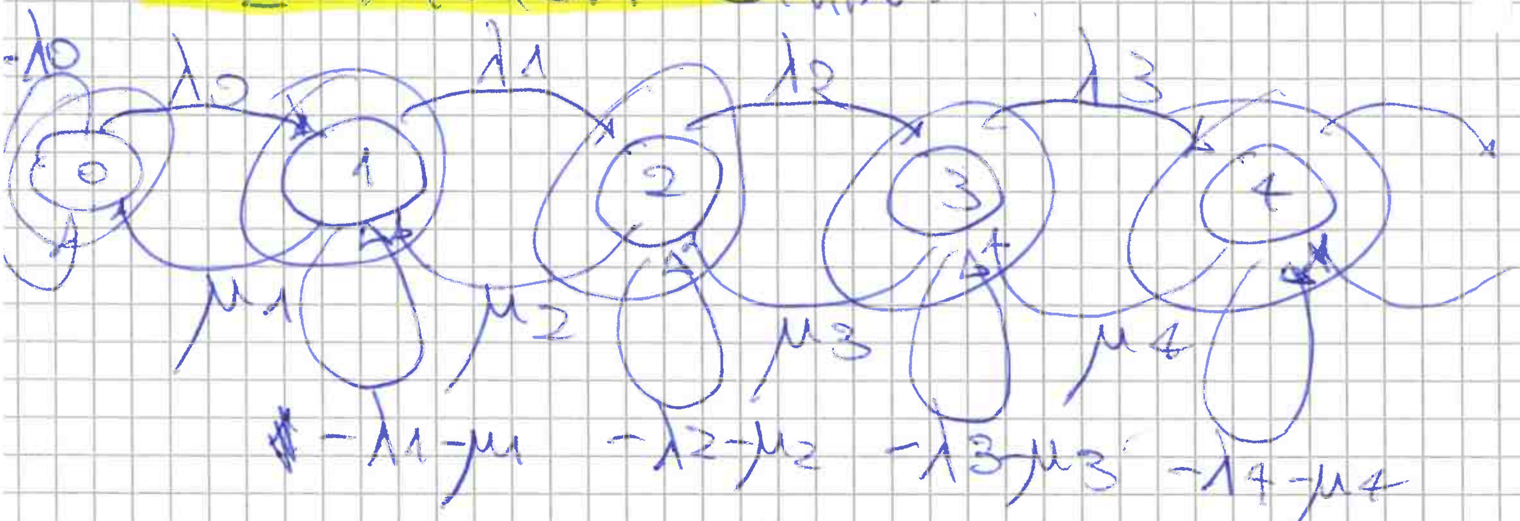
$$p_0 = \frac{1}{1 - \frac{b}{d}}$$

(like M/M/1)

$$\Rightarrow \boxed{p_i = \left(\frac{b}{d}\right)^i \left(1 - \frac{b}{d}\right)}$$

GEOMETRIC BEHAVIOUR (DECAY)

24) HOMOGENEOUS BIRTH-DEATH CONTINUOUS-TIME MARKOFF CHAIN



$$\underline{V} = [v_{ij}] \qquad v_{ij} = \begin{cases} \lambda_i & j = i+1 \\ \mu_1 & j = i-1 \\ -\lambda_i - \mu_i & j = i \neq 0 \\ -\lambda_0 & j = i = 0 \\ o(\Delta t) & \text{elsewhere} \end{cases}$$

Again, 3-DIAGONAL MATRIX.

a) FCP for TRANSIENT ANALYSIS:

ENTERING LEAVING

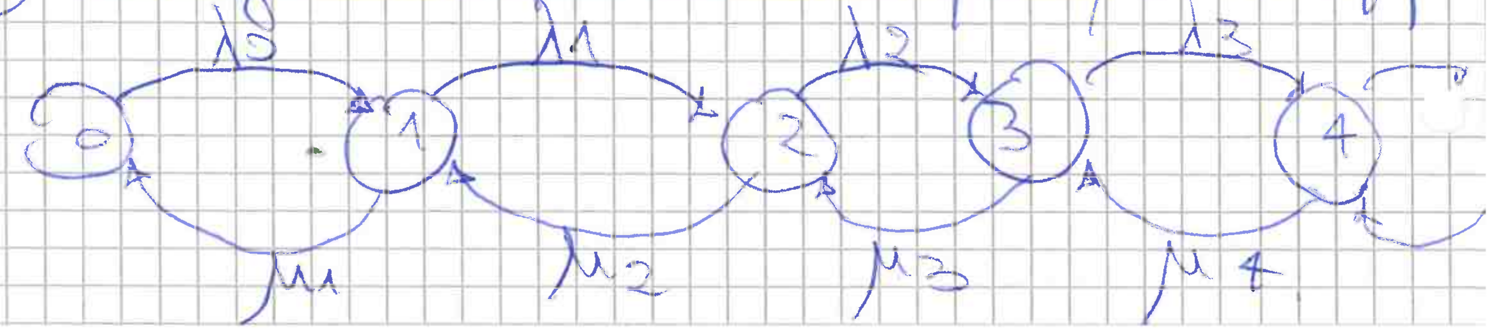
$$\frac{d}{dt} p_0(t) = p_1(t)\mu_1 - p_0(t)\lambda_0$$

$$\frac{d}{dt} p_1(t) = p_2(t)\mu_2 - p_1(t)\lambda_1$$

ENTERING ENTERING

$$\frac{d}{dt} p_i(t) = p_{i+1}(t)\mu_{i+1} + p_{i-1}(t)\cdot\lambda_{i-1} - p_i(t)(\lambda_i + \mu_i)$$

b) FCP for STEADY-STATE ANALYSIS (STATIONARY)



$\lambda_0 \cdot p_0 = \mu_1 \cdot p_1 \Rightarrow p_1 = \frac{\lambda_0}{\mu_1} p_0$

$\lambda_1 \cdot p_1 = \mu_2 p_2$

$\lambda_2 \cdot p_2 = \mu_3 p_3$

$\Rightarrow p_2 = \frac{\lambda_1}{\mu_2} p_1 = \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} p_0$

$\Rightarrow p_3 = \frac{\lambda_2}{\mu_3} p_2 = \frac{\lambda_2}{\mu_3} \cdot \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} p_0$

$$\Rightarrow \boxed{p_i = \frac{\lambda_0 \cdot \lambda_1 \cdot \lambda_2 \cdots \lambda_{i-1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_i} p_0}$$

$$\Rightarrow \boxed{p_i = p_0 \cdot \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}}$$

By the normalization condition $\boxed{\sum_{i=0}^{+\infty} p_i = 1}$

$$\Rightarrow \sum_{i=0}^{+\infty} p_0 \cdot \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} = 1$$

$$p_0 + p_0 \sum_{i=1}^{+\infty} \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} = 1$$

$$p_0 \cdot \left[1 + \sum_{i=1}^{+\infty} \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} \right] = 1$$

$$\boxed{p_0 = \frac{1}{1 + \sum_{i=1}^{+\infty} \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}}}$$

$$\boxed{p_0 = \frac{1}{1 + \sum_{i=1}^{+\infty} \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}}}$$

SERIES DIVERGES

If $\frac{\lambda_j}{\mu_{j+1}} > 1$

All STATES are NULL-RECURRENT

$\Downarrow$

$p_0 = 0$

SERIES CONVERGES

If $\frac{\lambda_j}{\mu_{j+1}} < 1$

$\Rightarrow \lambda_j < \mu_{j+1}$

$\Downarrow$

~~A~~ All STATES are POSITIVE RECURRENT

$\Downarrow$

Chain is ERGODIC

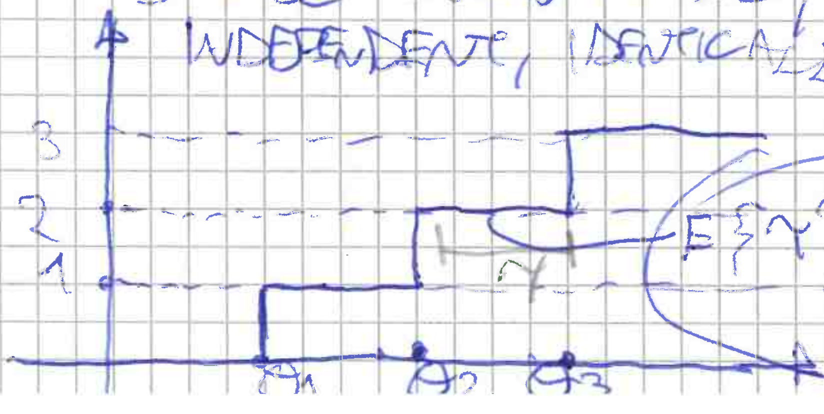
$0 < p_0 < 1$

(25) PURE-BIRTH MACHINE as a POISSON R.V.

~~Consider a POISSON / COUNTING PROCESS~~

Consider a PURE-BIRTH MACHINE, where:

- ARRIVALS are EXPONENTIALLY-DISTRIBUTED & INDEPENDENT, IDENTICALLY DISTRIBUTED.

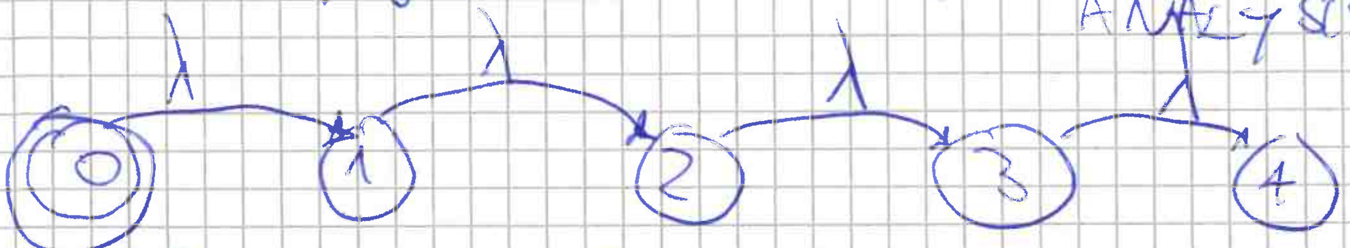


$$E\{Y\} = \frac{1}{\lambda}$$

RATE of ARRIVALS $= \lambda = \lambda_i$ (always the same!)

INTER ARRIVAL - RATE $= \dfrac{1}{\lambda}$

$\Rightarrow$ Apply FCP to it for TRANSIENT ANALYSIS:



To:

$$\begin{cases} \frac{d}{dt} p_0(t) = -\lambda p_0(t) \\ \frac{d}{dt} p_1(t) = -\lambda p_1(t) + \lambda p_0(t) \\ \frac{d}{dt} p_2(t) = -\lambda p_2(t) + \lambda p_1(t) \\ \vdots \\ \frac{d}{dt} p_i(t) = -\lambda p_i(t) + \lambda p_{i-1}(t) \end{cases}$$

Cannot solve ∞

However we can still solve by applying $\mathcal{L}$

$$\begin{cases} \frac{d}{dt} p_0(t) = -\lambda p_0(t) \\ \underline{P}(0) = (1\ 0\ 0\ 0\ 0 \ldots) \end{cases}$$

$s \cdot p_0(s) - p_0(0) = -\lambda p_0(s)$

$p_0(s) \cdot (s+\lambda) = 1$

$\Rightarrow p_0(s) = \dfrac{\lambda}{s+\lambda} \Rightarrow p_0(t) = e^{-\lambda t} \cdot u(t)$

$\Rightarrow$ Now take $\frac{d}{dt} p_i(t)$ for $i \neq 0$

$$\frac{d}{dt} p_i(t) = \lambda p_{i-1}(t) - \lambda p_i(t)$$

$$s \cdot p_i(s) - \cancel{p_i(0)} = \lambda \, p_{i-1}(s) - \lambda \, p_i(s)$$

$$p_i(s) \cdot (s + \lambda) = \lambda \, p_{i-1}(s)$$

$$\boxed{p_i(s) = \frac{\lambda}{(s+\lambda)} p_{i-1}(s)}$$

Take $i = 1$:

$$p_1(s) = \frac{\lambda}{s+\lambda} p_0(s) \qquad = \frac{1}{s+\lambda}$$

$$\Rightarrow p_1(s) = \frac{\lambda}{(s+\lambda)^2}$$

$$\Rightarrow p_2(s) = \frac{\lambda}{(s+\lambda)} \cdot p_1(s) = \frac{\lambda}{(s+\lambda)} \cdot \frac{\lambda}{(s+\lambda)^2} =$$

$$\boxed{p_2(s) = \frac{\lambda^2}{(s+\lambda)^3}}$$

$\Rightarrow$ By INDUCTION:

$$\boxed{p_n(s) = \frac{\lambda^n}{(s+\lambda)^{n+1}}}$$

We know:

$$\frac{1}{(s+\lambda)^{n+1}} \xrightarrow{\mathcal{L}^{-1}} \frac{t^n}{n!} \cdot e^{-\lambda t}$$

$$\Rightarrow P_n(t) = \frac{\lambda^n \cdot t^n}{n!} \cdot e^{-\lambda t} = \frac{(\lambda t)^n}{n!} \cdot e^{-\lambda t} = \frac{(\Lambda)^n}{n!} \cdot e^{-\lambda t}$$

POISSON DISTRIBUTION

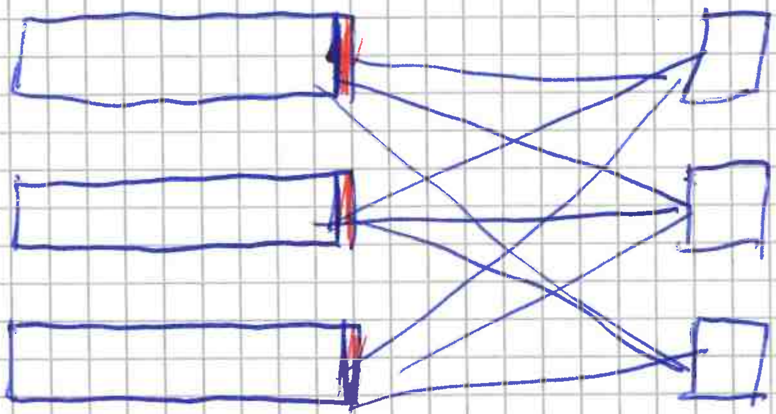
Probability of finding the process in state n ~~over~~ always over time.

[P. of having n arrivals in interval $[0,t]$]

⇒ Generally we have such distribution for arrivals

26 PACKET-SWITCHING ARCHITECTURES

① INPUT QUEUEING: A buffer is located at every INPUT LINE.

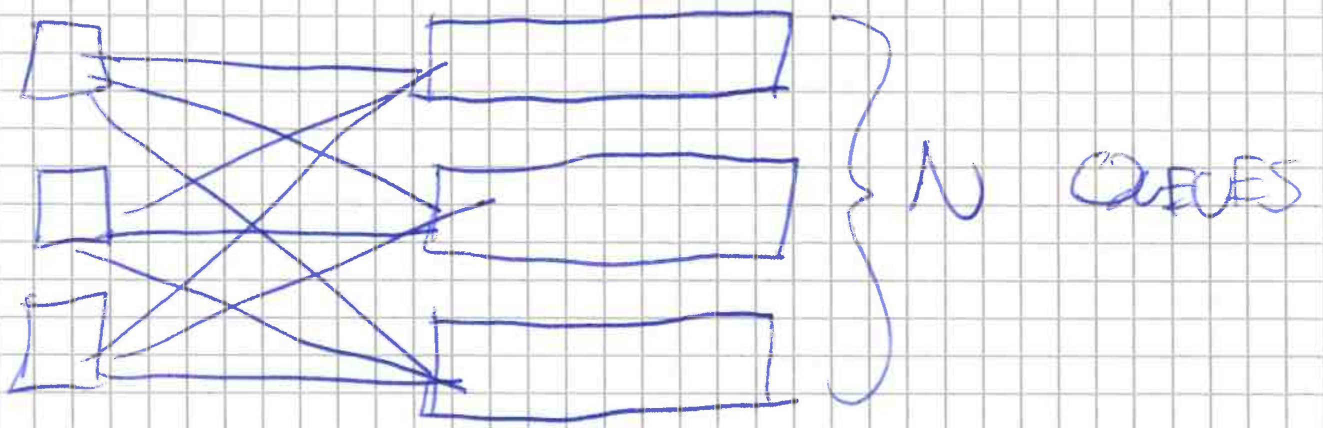


HOL (Head-of-Line) problem: where packets ~~[illegible]~~ transmitted one after the other are "queued" even if they have different destinations.

② OUTPUT QUEUEING:

Queues are located at every OUTPUT LINE

⇒ No longer BLOCKING ~~[illegible]~~ with different destinations-packets.



N x SPEEDUP FACTOR (N = # VIRTUAL BUFFERS)

(3) VIRTUAL OUTPUT QUEUEING.

Use N "virtual" buffer for INPUT QUEUE (~~INPUT~~ AT BUFFER)
⇒ SCHEDULING PROBLEM.
"Which queue ought to be served?"

⇒ Such architectures are used within packet switches. [With fixed-size packets, we can best analyze a system]

GEOMETRIC ARRIVALS, GEOMETRIC SERVICE

(27) GEO / GEO / 1 QUEUE.

It is a discrete-time queue used to model fixed-size packets arrivals to a system (Ex: ATM), where we have BERNOULLI R.V.S arriving (have a cell or not have it BUSY)

GEOMETRIC ARRIVALS & GEOMETRIC SERVICE.

$P\{\text{busy slot}\} = \alpha$ ARRIVAL

$P\{\text{empty slot}\} = 1 - \alpha$

$E\{A\} = \frac{1}{\alpha}$

~~VIRTUAL~~ $P\{SERVICE\} = \beta$ SERVICE

$P\{NO\ SERVICE\} = 1 - \beta$

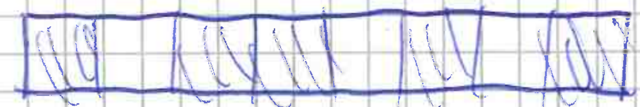
$$E\{S\} = \frac{1}{\beta}$$

GEOMETRIC INTERARRIVAL TIME:

$\Rightarrow$ We have already shown that

$P\{A = k\} = \alpha(1-\alpha)^{k-1}$ [k ARRIVALS]

$$E\{A\} = \frac{1}{1-\alpha}$$



GEOMETRIC INTER-SERVICE TIME:

$P\{B = k\} = \beta(1-\beta)^{k-1}$ [k SERVICES]

$$E\{B\} = \frac{1}{1-\beta}$$

STATE = # CUSTOMERS in the QUEUE

TRANSITIONS:

$\boxed{0} \xrightarrow{\alpha} \boxed{1}$ ONE ARRIVAL

$\boxed{1} \xrightarrow{\alpha(1-\beta)} \boxed{2}$ ONE ARRIVAL, NO SERVICE

$\boxed{1} \xrightarrow{\beta(1-\alpha)} \boxed{0}$ NO ARRIVAL, BUT SERVICE

$\boxed{1} \xrightarrow{\alpha\beta} \boxed{1}$ NO ARRIVAL, NO SERVICE

OR

1 ARRIVAL, 1 SERVICE

PROBABILITY of REMAINING in the SAME STATE

① NO ARRIVALS & NO SERVICE

ARRIVAL

$$1-\left[\alpha(1-\beta)+\beta(1-\alpha)\right]$$

$$=1-\alpha+\alpha\beta-\beta+\alpha\beta$$

$$=1-\alpha-\beta+2\alpha\beta$$

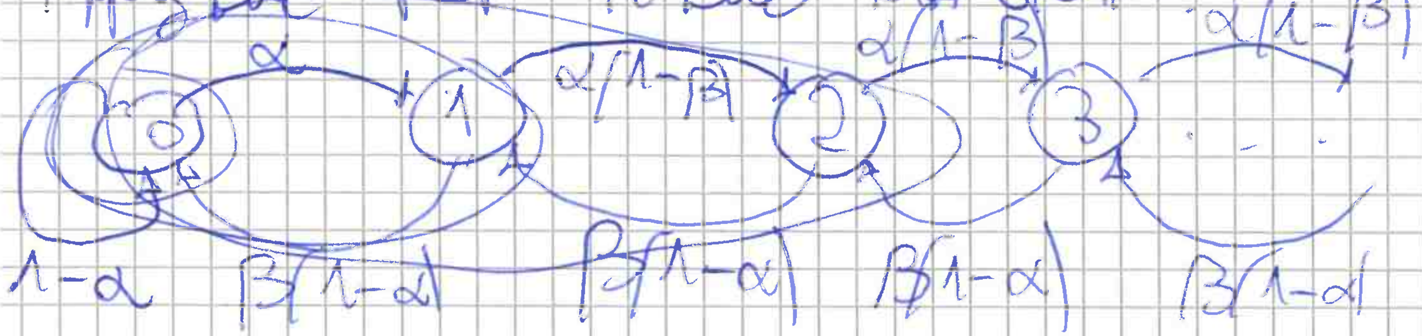
② 1 SERVICE, 1 ARRIVAL ~~& NO~~

$$\underbrace{\alpha\beta}_{\text{SERVICE \& ARRIVAL}}+\underbrace{(1-\alpha)\cdot(1-\beta)}_{\text{NO SERVICE, NO ARRIVAL}}$$

~~ARRIVAL~~ OR

③ ERGODICITY CONDITION;

Apply the FCP to the DIAGRAM:



If $\alpha=1$, $\beta=0$ ⇒ NOT PERIODIC! (NULL-RECURRENT?)
⇒ APERIODIC (A simple limit)

$0<\alpha<1$, $0<\beta<1$ ⇒ PERIODIC with PERIOD = 2

Apply the BALANCE EQUATIONS through the FCP;

$$\cancel{\alpha p_0 = \beta(1-\alpha) p_1 \Rightarrow p_1 = \frac{\alpha}{\beta(1-\alpha)} p_0}$$

$$\cancel{(1-\beta)\alpha p_1 = \beta(1-\alpha) p_2 \Rightarrow p_2 = \frac{\alpha(1-\beta)}{\beta(1-\alpha)} p_1}$$

$$\alpha p_0 = \beta(1-\alpha) p_1 \Rightarrow p_1 = \frac{\alpha}{\beta(1-\alpha)} p_0$$

$$\alpha(1-\beta) p_1 = \beta(1-\alpha) p_2 \Rightarrow p_2 = \frac{\alpha(1-\beta)}{\beta(1-\alpha)} p_1$$

$$\alpha(1-\beta) p_2 = \beta(1-\alpha) p_3$$

$$p_2 = \frac{\alpha(1-\beta)\cdot\alpha}{\beta(1-\alpha)\cdot\beta(1-\alpha)} p_0$$

$$\Rightarrow p_3 = \frac{\alpha(1-\beta)}{\beta(1-\alpha)} p_2 = \left[\frac{\alpha}{\beta(1-\alpha)}\right]^2 (1-\beta) p$$

$$\Rightarrow p_3 = \left[\frac{\alpha}{\beta(1-\alpha)}\right]^3 (1-\beta)^2 p_0$$

In general, for i:

$$p_i = \frac{\alpha(1-\beta)}{\beta(1-\alpha)} p_{i-1}$$

$$\Rightarrow p_i = \left[\frac{\alpha}{\beta(1-\alpha)}\right]^i \cdot (1-\beta)^{i-1} \cdot p_0$$

$$\boxed{p_i = \left[\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right]^{i-1} \cdot \left(\frac{\alpha}{\beta(1-\alpha)}\right) p_0}$$

$$\Rightarrow p_i = \left[\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right]^i \frac{p_0}{(1-\beta)}$$

Apply the NORMALIZATION CONDITION $\boxed{\sum_{i=0}^{+\infty} p_i = 1}$

$$p_0 + \sum_{i=1}^{+\infty}\left[\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right]^i \frac{p_0}{(1-\beta)} = 1$$

$$p_0 + \frac{p_0}{1-\beta}\sum_{i=1}^{+\infty}\left[\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right]^i = 1$$

If this series CONVERGES $\Rightarrow$ All states are POSITIVE RECURRENT & Chain is <u>ERGODIC</u>.

IF $\dfrac{\alpha(1-\beta)}{\beta(1-\alpha)} < 1$

$$\alpha(1-\beta) < \beta(1-\alpha)$$

$$\alpha - \cancel{\alpha\beta} < \beta - \cancel{\alpha\beta}$$

ERGODICITY CONDITION $\Rightarrow \alpha < \beta$

28) SOLVING THE (FORWARD) CHAPMAN-KOLMOGOROFF EQUATION for a PURE-BIRTH HCTMC

$$0 \xrightarrow{\lambda_0} 1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} 3 \xrightarrow{\lambda_3} \cdots \xrightarrow{\lambda_{n-2}} n-1 \xrightarrow{\lambda_{n-1}} n \xrightarrow{\lambda_n}$$

FORWARD KOLMOGOROFF EQUATION:

$$\begin{cases} \dfrac{d}{d\tau} \underline{\underline{H}}(\tau) = \underline{\underline{H}}(\tau) \cdot \underline{\underline{V}} \\ \underline{\underline{H}}(0) = \underline{\underline{I}} \end{cases}$$

In SCALAR FORM, this is:

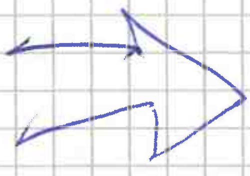
$$\frac{d}{d\tau} h_{ij}(\tau) = \underbrace{\sum_{k \in S} h_{ik}(\tau)\, v_{kj}}$$

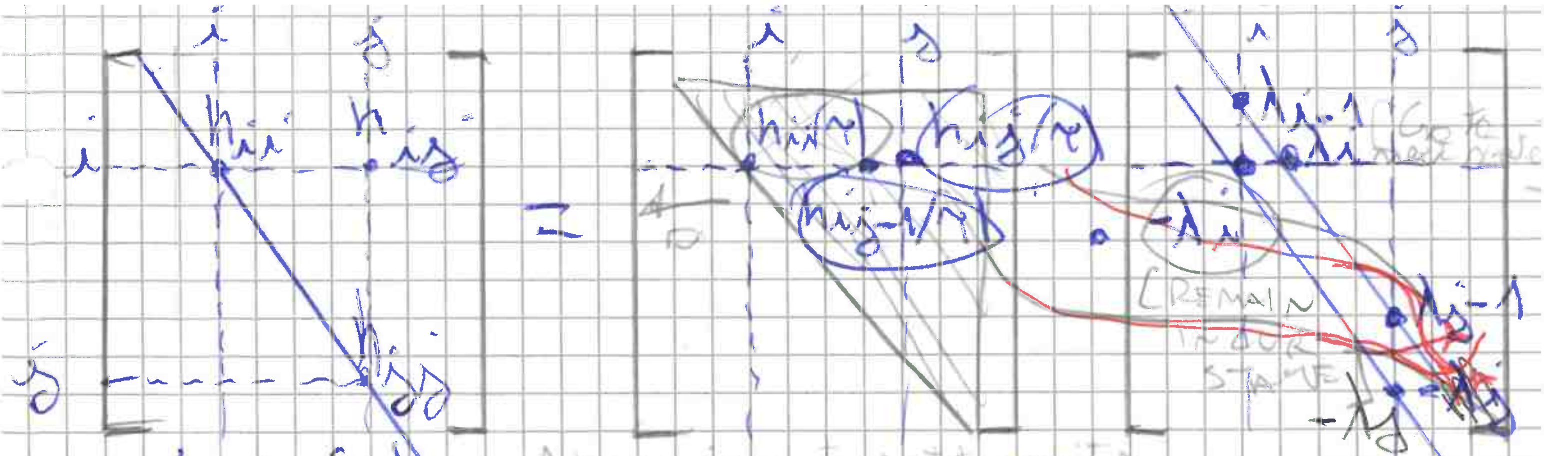
Where: $\sum_{j \in S} h_{ij}(\tau) = 1$

and $\underline{\underline{H}}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$

$$\underline{\underline{H}}(0) = [h_{ij}(0)] = \begin{cases} h_{ij}(0) = 1 & \forall i = j \\ h_{ij}(0) = 0 & \forall i \neq j \end{cases}$$

$$\underline{\underline{H}}(0) = \underline{\underline{I}}$$



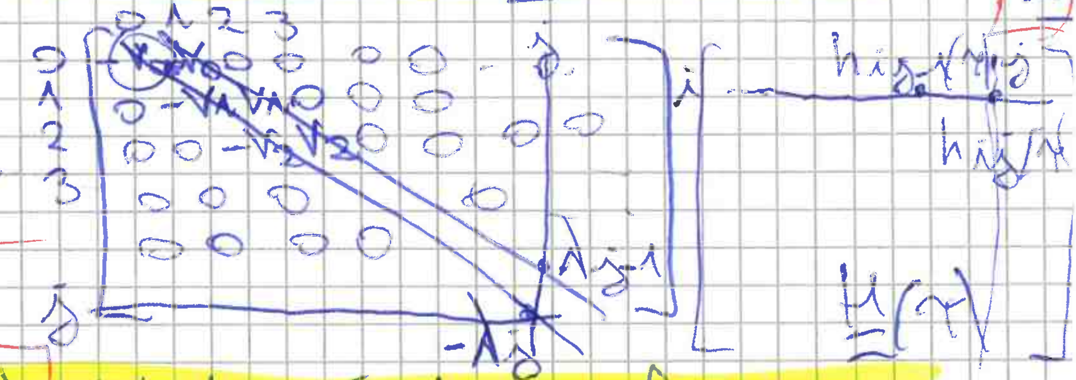


$$\frac{d}{d\tau}\underline{\underline{H}}(\tau) = \qquad \underline{\underline{H}}(\tau) \neq \text{TRIANGULAR} \qquad \underline{\underline{V}}\,[\text{BIDIAGONAL}]$$

↳ DIAGONAL MATRIX $\underline{\underline{H}}(\tau)$

NOW: BI-DIAGONAL MATRIX FOR $\underline{\underline{V}}$

$\underline{\underline{V}} =$



$\underline{\underline{H}}(\tau)$ TRIANGULAR MATRIX

(Never jump from i to a state

⇒ MATRIX MULTIPLICATION can be expressed as:

MAIN DIAGONAL: $\frac{d}{d\tau} h_{ii}(\tau) = -\lambda_i \cdot h_{ii}(\tau)$

(ROW i X COLUMN j)

OUT MAIN DIAGONAL: $\frac{d}{d\tau} h_{ij}(\tau) = \lambda_{j-1} \cdot h_{ij-1}(\tau) - \lambda_j \, h_{ij}(\tau)$

$\underline{\underline{H}}(0) = \underline{\underline{I}}$

(1) Solve the DIFF. EQUATION for the main diagonal (one unknown only!)

$$\frac{d}{dt} h_{ii}(\tau) = -\lambda_i \, h_{ii}(\tau)$$

$s \cdot h_{ii}(s) - h_{ii}(0) = -\lambda_i \cdot h_{ii}(s)$

$h_{ii}(0) = 1$ because $\underline{\underline{H}}(0) = \underline{\underline{I}}$

$h_{ii}(s)(s+\lambda_i) = 1$

$$\Rightarrow h_{ii}(s) = \frac{1}{s+\lambda_i}$$

$$\boxed{\Rightarrow h_{ii}(\tau) = e^{-\lambda_i \cdot \tau} \qquad \underline{\underline{H}}(\tau)} \Rightarrow \text{COMPONENTS ON MAIN DIAGONAL ARE EXPONENTIAL FORM}$$

probability to remain in state decays over time

$\underline{P}(t+\tau) = \underline{P}(t) \cdot \underline{\underline{H}}(\tau)$

$$\boxed{\underline{P}(t) = \underline{P}(0) \cdot \underline{\underline{H}}(t)}$$

TRANSIENT BEHAVIOUR

Counter initial value i_0 at time 0.

ANY POSSIBLE PATTERN

$\underline{P}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to i_0$

in a counting process (pure birth) $\Rightarrow$ counter's initial value

Remember: $P_j(t) = \sum_{k \in S} P_k(t) \cdot h_{kj}(t)$

ROW COLUMN

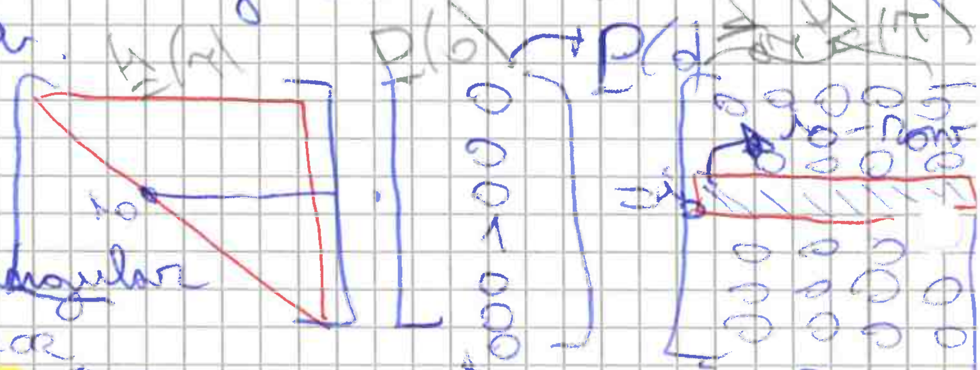
P(0)

We will hence get:

$$\boxed{P_j(t) = h_{i_0 j}(t)} \quad \forall j$$

$\Rightarrow$ Only components left are the ones on the i_0-row.

$\underline{\underline{H}}(t) =$



INITIALLY: Triangular superior

AFTERWARDS: Only components at i_0-row

$\Rightarrow$ $\boxed{\underline{P}(E) = i_0\text{-ROW}}$

$$\underline{P}(E) = \begin{bmatrix} p_1(E) & p_2(E) & \dots & p_n(E) \end{bmatrix}$$

$$\begin{bmatrix} h_{i_0 1} & h_{i_0 2} & \dots & h_{i_0 n} \end{bmatrix}$$

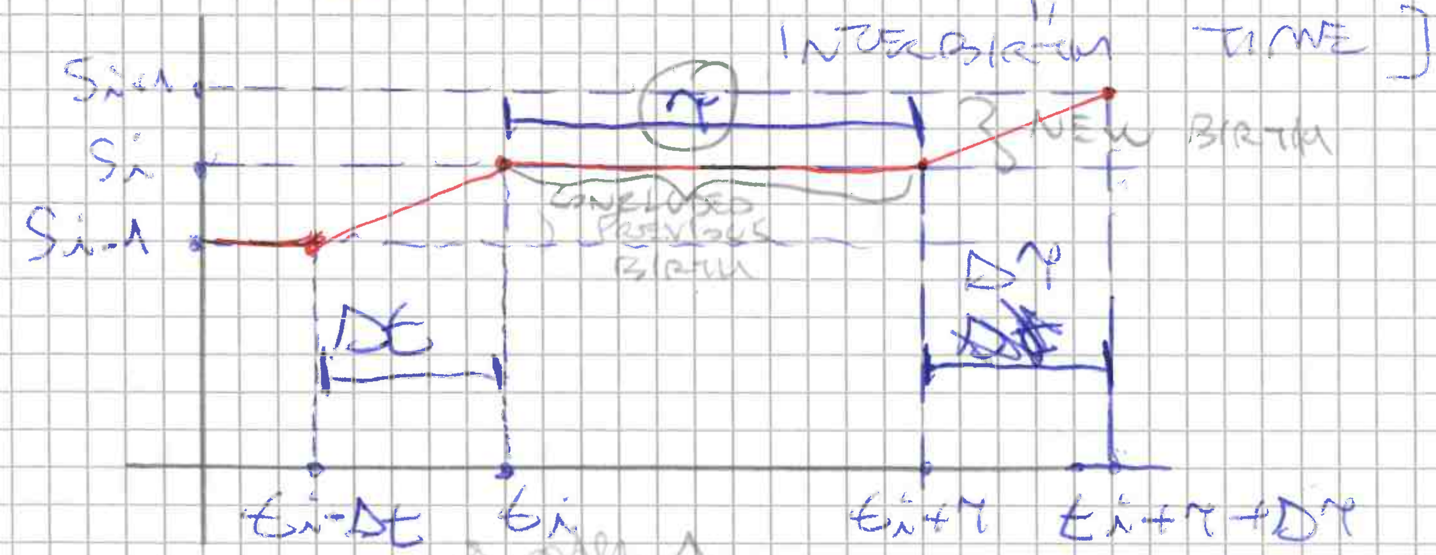
~~This is all we can say!~~

ANALYSIS stops here!

$\Rightarrow$ COMPLEX to evaluate components not on the MAIN DIAGONAL!

ORDER-1 INTERBIRTH TIME:

GOAL: Understand PDF of ORDER-1 INTERBIRTH time. [INTERARRIVAL TIME = INTERBIRTH TIME]



INTERARRIVAL: Time passing in-between two BIRTHS (i.e: two arrivals) (one birth and another one)

$\Rightarrow$ Precisely define interbirth time & find its DISTRIBUTION.

PROOF: Is the INTERBIRTH ~~INTERARRIVAL~~ TIME effectively EXPONENTIAL? $\Rightarrow$ find PDF for the INTERBIRTH time.

DENSITY FUNCTION I'm looking for $\rightarrow$ P. the r.v. ranging from τ to $\Delta\tau$

$$f_i(\tau)\cdot\Delta\tau = P\{X(t_i+\tau+\Delta\tau)=s_{i+1} \mid X(t_i+\tau)=s_i,\ X(t_i)=s_i,\ X(t_i-\Delta t)=s_{i-1}\}$$

B

$\Rightarrow$ Rewrite it using the BAYES THEOREM.

$$f_i(\tau)\cdot\Delta\tau = P\{X(t_i+\tau+\Delta\tau)=s_{i+1} \mid X(t_i+\tau)=s_i,\ \cancel{X(t_i)=s_i,\ X(t_i-\Delta t)=s_{i-1}}\}$$

$$\cdot P\{X(t_i+\tau)=s_i \mid X(t_i)=s_i,\ \cancel{X(t_i-\Delta t)}=s\ldots$$

$g_i(t)\,\Delta\tau = P\{\cancel{X}(t_i+\tau) = s_{i+1} \mid \cancel{X}(t_i+\tau) = s \ldots$

$\underbrace{P\{\cancel{X}(t_i+\tau) = s_i \mid \cancel{X}(t_i) = s_i\}}_{h_{ii}(\tau)}$ (REMAIN IN SAME STATE AFTER TIME τ)

NEW ARRIVAL $\lambda_i \Delta\tau$

$$\Rightarrow \boxed{g_i(\tau)\cancel{\Delta\tau} = \lambda_i \cdot \cancel{\Delta\tau} \cdot h_{ii}(\tau)}$$

$$\Rightarrow \boxed{g_i(\tau) = \lambda_i \cdot h_{ii}(\tau)}$$

(BECAUSE MEMORYLESS & I.I.D. ARRIVALS)

$$\Rightarrow \boxed{g_i(\tau) = \lambda_i \cdot e^{-\lambda_i \tau}\, u(\tau)}$$

EXPONENTIAL DISTRIBUTION of the FIRST-ORDER INTER-BIRTH TIME

(i.e.: SOJOURN TIME | for a LIFETIME)

n-ORDER INTERBIRTH
Time passing between (n) births.

② ⓐ MOMENT GENERATING FUNCTION $M(s)$:

We know:

$$E\{X\} = \int_0^{+\infty} x \cdot f_X(x)\, dx$$ EXPECTED VALUES DEFINITION

$$\boxed{M(s) = E\{e^{sX}\} = \int_0^{+\infty} f(x) \cdot e^{sx}\, dx}$$

From the $\mathcal{L}$-Transform, we know:

$$\mathcal{L}\{f(x)\} = \int_0^{+\infty} f(x) \cdot e^{-sx}\, dx$$

If $s = -s \Rightarrow$ You get the MOMENT GENERATING FUNCTION

$$\boxed{M(s) = \mathcal{L}\{f(x)\}}$$

a) r-ORDER MOMENT:

$$\boxed{E\{x^r\} = \frac{d^r}{ds^r} M(s) \Big|_{s=0}}$$

ⓑ DEFINITION of VARIANCE:

$$\boxed{VAR\{x\} = E\{x^2\} - (E\{x\})^2}$$

ⓒ COEFFICIENT of VARIATION:

$$CV = \frac{\sqrt{\sigma_v^2}}{E\{x\}}$$

VARIANCE

STANDARD DEVIATION

d) $M(s)$, $E\{x\}$, $E\{x^2\}$ for the EXP. PDF

$$\boxed{f(x) = \lambda \cdot e^{-\lambda x} \cdot u(x)}$$

$$M(s) = \mathcal{L}\{\lambda \cdot e^{-\lambda x} \cdot u(x)\} = \frac{\lambda}{s-\lambda}$$

$$E\{x\} = \frac{d}{ds}\frac{\lambda}{s-\lambda} = \frac{\lambda}{(s-\lambda)^2}\bigg|_{s=0} = \frac{\cancel{\lambda}}{\lambda^{\cancel{2}}} = \frac{1}{\lambda}$$

$$E\{x^2\} = \frac{d}{ds^2}\frac{\lambda}{s-\lambda} = \frac{2\lambda}{(s-\lambda)^3}\bigg|_{s=0} = \frac{2\cancel{\lambda}}{\lambda^3} = \frac{2}{\lambda^2}$$

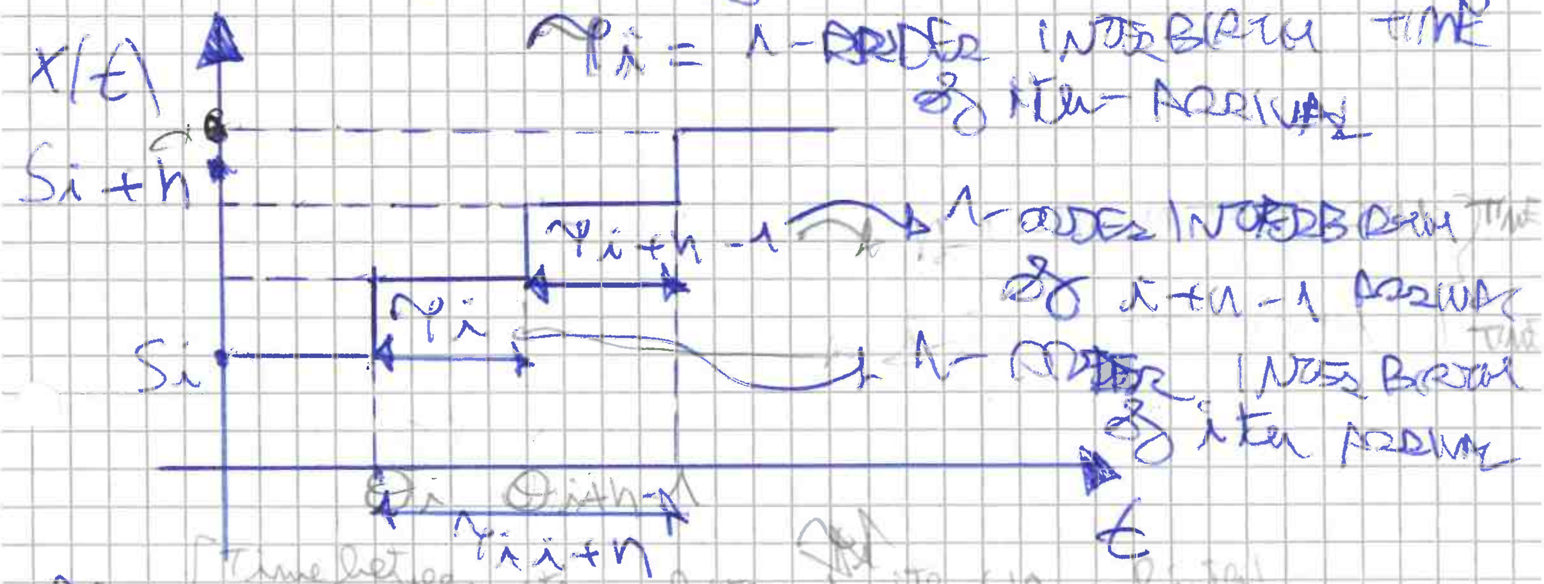
$$VAR\{x\} = E\{x^2\} - (E\{x\})^2$$

$$\boxed{VAR\{x\}} = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda^2}\right) = \boxed{\frac{1}{\lambda^2}}$$

$$CV = \frac{\sqrt{\frac{1}{\lambda^2}}}{E\{x\}} = \frac{\frac{1}{\cancel{\lambda}}}{\frac{1}{\cancel{\lambda}}} = 1$$

30) PDF of the n-order INTERBIRTH TIME as an Erlang-n DISTRIBUTED R.V.

Consider a process with n-many ORDER-1 INTERBIRTHS (Sum of n 1-ORDER INTERBIRTHS)



τ_i is EXPONENTIALLY-DISTRIBUTED (1-ORDER INTERBIRTH TIME) of ith ARRIVAL

$$f(\tau) = \lambda \cdot e^{-\lambda \tau} \mu(\tau)$$

[n-many ARRIVALS]

From τ_i to τ_{i+n-1} there are n CONTRIBUTIONS:

ARRIVAL RATE TRANSITION RATE

n-ORDER PDF

$$\boxed{f_{i,i+n}(t) = \lambda_{i+n-1} \cdot h_{i,i+n-1}(t)}$$

$e^{-\lambda_i t}\mu(t)$

$$\boxed{\underbrace{\tau_{i,i+n}}_{\text{n-ORDER INTERBIRTH TIME}} = \sum_{k=0}^{n-1} \tau_{i+k}}$$

$f_i(t) = \lambda_i \cdot h_i(t)$

Take $E\{\cdot\}$:

$$E\{\tau_{i,i+n}\} = \sum_{k=0}^{n-1} E\{\tau_{i+k}\} = \sum_{k=0}^{n-1} \frac{1}{\lambda_{i+k}}$$

Since τ_{i+k} are independent from one another.

$$\cancel{VAR\{\tau_{i,i+n}\} = \sum_{k=0}^{n-1} \frac{1}{\lambda^2_{i+k}}}$$

$$VAR\{T_{i\,i+n}\} = \sum_{K=0}^{n-1} \frac{1}{\lambda_{i+K}^2}$$

$f_{i+K}(\tau) = \lambda_{i+K}\, e^{-\lambda_{i+K}\tau}$

$F_{i+K}(s) = \dfrac{\lambda_{i+K}}{s+\lambda_{i+K}}$

~~⇒ Taking the $\mathcal{L}$-Transform~~

⇒ The ~~n-order~~ PDF of n-order interarrival is hence given by:

PDF of n-th interarr.

CONVOLUTION of the single independent PDFs

$$f_{i\,i+n}(\tau) = f_i(\tau) \circledast f_{i+1}(\tau) \circledast \dots \circledast f_{i+n-1}(\tau)$$

⇒ Taking the $\mathcal{L}$-transform of the convolution

$$F_{i\,i+n}(s) = \prod_{K=0}^{n-1} \frac{\lambda_{i+K}}{s+\lambda_{i+K}}$$

In the special case of a POISSON PROCESS:

$$\lambda_{i+K} = \lambda$$

$$F_{i\,i+n}(s) = \prod_{K=0}^{n-1} \frac{\lambda}{s+\lambda} = \frac{\lambda^n}{(s+\lambda)^n}$$

$\dfrac{\lambda}{(s+\lambda)^{n+1}} \rightarrow \dfrac{t}{n!} \cdot e^{-\lambda t}$

We know:

$$\frac{1}{(s+\lambda)^n} \xleftrightarrow{\mathcal{L}} \frac{t^{n-1}}{(n-1)!}\, e^{-\lambda t}$$

n-ERLANG DISTRIBUTION

$$\Rightarrow f_{i\,i+n}(t) = \frac{\lambda^n \cdot t^{n-1}}{(n-1)!} \cdot e^{-\lambda t}$$

$$= \lambda \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \cdot e^{-\lambda t}$$

⇒ We hence have found the ERLANG-n DISTRIBUTION

$$f_{ii+n}(t) = \frac{\lambda \cdot (\lambda t)^{n-1}}{(n-1)!} \cdot e^{-\lambda t} \, \mu(t)$$

ⓐ $E\{\tau_{ii+n}\} = \frac{n}{\lambda}$ (n · EXP. distribution's mean)

$VAR\{\tau_{ii+n}\} = \frac{n}{\lambda^2}$ (n · EXP. distribution's variance)

$$CV = \frac{\sqrt{\frac{n}{\lambda^2}}}{\frac{n}{\lambda}} = \frac{\frac{\sqrt{n}}{\lambda}}{\frac{n}{\lambda}} = \cancel{\frac{\sqrt{n}}{n}} \frac{\sqrt{n}}{\sqrt{n}} = \frac{1}{\sqrt{n}}$$

ⓑ Γ-DISTRIBUTION: $\Gamma(\nu) = (\nu - 1)!$

$$f_\nu(\tau) = \frac{\lambda \cdot (\lambda \tau)^{\nu-1}}{\Gamma(\nu)} \cdot e^{-\lambda \tau} \, \mu(\tau)$$

$E\{\tau\} = \frac{\nu}{\lambda}$ $\quad VAR\{\tau\} = \frac{\nu}{\lambda^2}$

NB: The Γ-DISTRIBUTION is a more general distribution than the ERLANG-n ou

31 DISCRETE-TIME BERNOULLI PROCESS

• **BERNOULLI R.V.**: R.V. that can only have a 0 or 1 value.

BERNOULLI DISTRIBUTION: Distribution characterizing a BERNOULLI R.V. (PMF).

$P\{X=0\} = \boxed{q} = 1-p$ = ~~SUCCESS~~ FAILURE

$P\{X=1\} = \boxed{p}$ = ~~FAILURE~~ SUCCESS.

Where X is a BERNOULLI R.V.

• GEOMETRIC INTERARRIVAL TIME

BERNOULLI PROCESS

Sum of n-independent BERNOULLI R.Vs.

BINOMIAL DISTRIBUTION

Distribution characterizing a Bernoulli Process (PMF)

$$\boxed{P\{X_n = i\} = P_i(n) = \binom{n}{i} p^i \cdot q^{n-i}}$$

$$E\{x^2\} = \frac{d}{dz^2}(q+pz)^N + \frac{d}{dz}(q+pz)^N \Big|_{z=1}$$

$$= \frac{d}{dz} N \cdot p \cdot (q+pz)^{N-1} + Np$$

$$= Np(N-1) \cdot p \cdot (q+pz)^{N-1} + Np$$

$$= N \cdot p^2 \cdot (N-1) + Np$$

$$\cancel{= Np \cdot (N} = N^2p^2 - Np^2 + Np$$

$$= Np \cdot (Np \underbrace{-p+1}_{q})$$

$$\boxed{E\{x^2\} = Np \cdot (Np+q)}$$

$$\Rightarrow VAR\{x\} = E\{x^2\} - (E\{x\})^2$$

$$= Np(Np+q) - (Np)^2$$

$$= \cancel{N^2p^2} + Npq \cancel{- N^2p^2}$$

$$\Rightarrow VAR\{x\} = N \cdot pq$$

(b) For a Bernoulli R.V.: Θ :

$$E\{\Theta\} = 0 \cdot q + 1 \cdot p = p$$

$\Rightarrow$ We can hence see that:

$$E\{x\} = \sum_{i=1}^{N} E\{\Theta_i\} = N \cdot p$$

$$E\{\theta^2\} = 0 \cdot q + 1^2 \cdot p = p$$

$$VAR\{\theta\} = E\{\theta^2\} - (E\{\theta\})^2$$

$$= p - p^2 = p\underbrace{(1-p)}_{q} = p \cdot q$$

$$VAR\{t\} = \sum_{i=1}^{N} VAR\{\theta_i\} = N \cdot p \cdot q$$

(32) AXIOMATIC DEFINITION of a POISSON PROCESS:

The Poisson Process consists of a COUNTING PROCESS with UNITARY INCREASES
⇒ A counting process with UNITARY INCREASES is a Poisson Process if the following (4) conditions hold:

(a) PROBABILITY of one arrival in the infinitesimal time interval dt is:

$$P\{1 \text{ arrival in } dt\} = \lambda dt + \underbrace{o(dt)}_{}$$

$\lambda_i = \lambda$ INF. of higher order

(b) PROBABILITY of NO (0) ARRIVALS in dt is:

$$P\{0 \text{ arrivals in } dt\} = 1 - \lambda dt + o(dt)$$

(c) Disjoint intervals are characterized by INDEPENDENT EVENTS (i.e.: the memoryless property holds)

(d) $P\{2 \text{ or more arrivals in } dt\} = o(dt)$

$\Rightarrow P\{0 \text{ arrivals in } dt\} + P\{1 \text{ arrival in } dt\}$

FIGURE THIS OR THIS

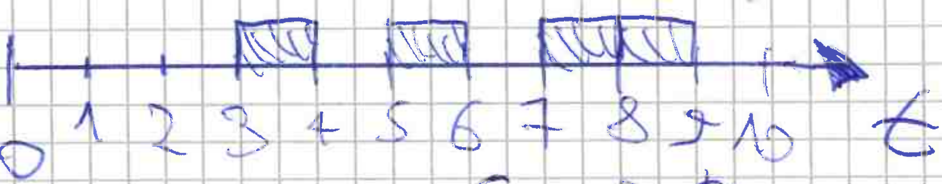
$+ P\{2 \text{ arrivals in } dt\}$

OR MORE

$= 1$

33 POISSON PROCESS AS A LIMITING CASE OF A DISCRETE-TIME BERNOULLI PROCESS (FOR $M \to \infty$ ATTEMPTS)

Consider a period $(0, t]$ sub-divided into small SUB-INTERVALS. The PROBABILITY to be in a certain SUB-INTERVAL is BINOMIAL (BINOMIAL DISTRIBUTION)



$M = \#$ SLOTS

t = Sum of the DURATION of the different slots intervals Δt $\Rightarrow (0, t)$

t = OVERALL TIME

$$\boxed{\Delta t = \frac{t}{M}}$$

→ Duration of one mini-interval (SLOT)

$p = \lambda \cdot \Delta t$ = P. to have an arrival in Δt

$q = 1 - \lambda \Delta t$ = P. not to have an arrival in Δt

$$\Rightarrow \boxed{P_i(t) = P_m(M)}$$ = P. to have M slots occupied (STATE PROBABILITY)

$$P_m(M) = \binom{M}{m} (\lambda \Delta t)^m \cdot (1 - \lambda \Delta t)^{M-m}$$

$\Delta t = \frac{t}{M}$

$p = \lambda \Delta t$

$q = 1 - \lambda \Delta t$

$$= \frac{M!}{m!\,(M-m)!} \left(\lambda \frac{t}{M}\right)^m \left(1 - \lambda \frac{t}{M}\right)^{M-m}$$

SAVE TERMS before $(M-m)!$

$$= \frac{M(M-1)(M-2)(M-3)\dots(M-m-1)}{m!\,(M-m)!} \left(\lambda \frac{t}{M}\right)^m \left(1 - \lambda \frac{t}{M}\right)^{M-m}$$

$$\frac{M(M-1)(M-2)\dots(M-m+1)\cdot\cancel{(M-m)!}}{\cancel{(M-m)!}}$$

$\lim_{M \to \infty}$

m terms

$$\Rightarrow \lim_{M \to \infty} \underbrace{\frac{M \cdot (M-1) \dots (M-m+1)}{M^m}}_{\to 1} \frac{(\lambda t)^m}{m!} \underbrace{\frac{\left(1 - \frac{\lambda t}{M}\right)^M}{\left(1 - \frac{\lambda t}{M}\right)^m}}_{= e^{-\lambda t}}$$

NUM. & DENOMINATOR have the same order of infinity

WELL-KNOWN LIMIT

$$\boxed{\frac{(\lambda t)^m}{m!} \cdot e^{-\lambda t} = \lim_{M \to \infty} p_m(M)} = p_m(t)$$

POISSON - DISTRIBUTION is a LIMITING CASE of a BERNOULLI DISTRIBUTION

$$\boxed{P_m(t) = \lim_{M \to \infty} P_m(M) = e^{-\lambda t} \cdot \frac{(\lambda t)^m}{m!}}$$

(3A) For X = Poisson R.V.

$G_X(z)$, $E\{x\}$, $E\{x^2\}$, $VAR\{x\}$,

$$G_X(z) = E\{z^X\} = \sum_{n=0}^{+\infty} p_n \cdot z^n$$

For X = Poisson R.V.

$$p_n = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

$\Lambda = \lambda t$.

$$\Rightarrow p_n = \frac{e^{-\Lambda} (\Lambda)^n}{n!}$$

$$\Rightarrow G_X(z) = \sum_{n=0}^{+\infty} e^{-\Lambda} \frac{(\Lambda z)^n}{n!} = e^{-\Lambda} \sum_{n=0}^{+\infty} \frac{(\Lambda z)^n}{n!}$$

$$e^{\Lambda z}$$

$$\Rightarrow G_X(z) = e^{-\Lambda} \cdot e^{\Lambda z} = e^{\Lambda (z-1)}$$

$$\Rightarrow \boxed{G_X(z) = e^{\Lambda (z-1)}}$$

$$E\{x\} = \frac{d}{dz} G_X(z)\Big|_{z=1} = \Lambda \cdot e^{\Lambda(z-1)}\Big|_{z=1}$$

$$\boxed{E\{x\} = \Lambda}$$

$$+\frac{d}{dz} G_X(z)\Big|_{z=1} \qquad +\frac{d}{dz} G_X(z)\Big|$$

$$E\{x^2\} = \frac{d^2}{dz^2} G_X(z)\Big|_{z=1} = \frac{d}{dz} \Lambda \cdot e^{\Lambda(z-1)}\Big|_{z=1}$$

$$= \Lambda \cdot \Lambda \cdot e^{\Lambda(z-1)}\Big|_{z=1} = \Lambda^2 + \Lambda = \Lambda(\Lambda+1)$$

$$\boxed{E\{x^2\} = \lambda(\lambda+1)}$$

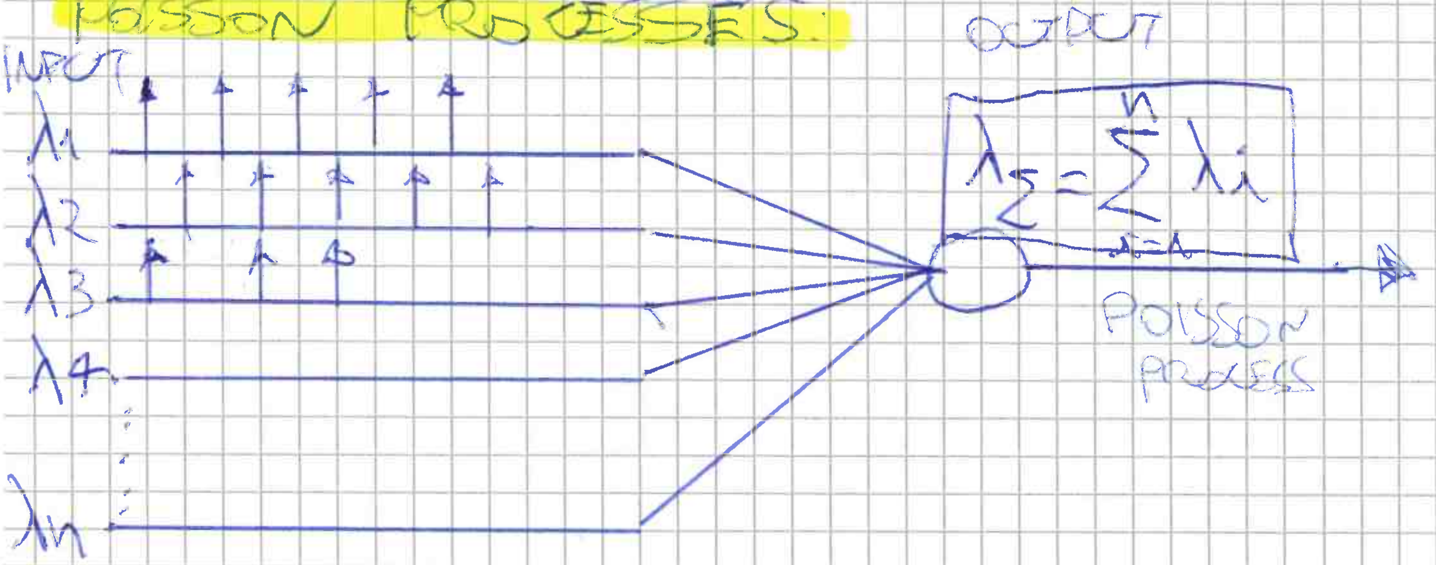
$$VAR\{x\} = E\{x^2\} - (E\{x\})^2$$

$$= \lambda(\lambda+1) - \lambda^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$\Rightarrow \boxed{VAR\{x\} = \lambda}$$

$$CV = \frac{\sqrt{\sigma^2}}{E\{x\}} = \frac{\sqrt{\lambda}}{\lambda} \cdot \frac{\sqrt{\lambda}}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}}$$

35 COMBINATION of n-independent POISSON PROCESSES.



$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the ARRIVAL RATES of n POISSON PROCESSES.

We want to show that

<u>INPUT</u>: n-many POISSON PROCESSES (independent from one another

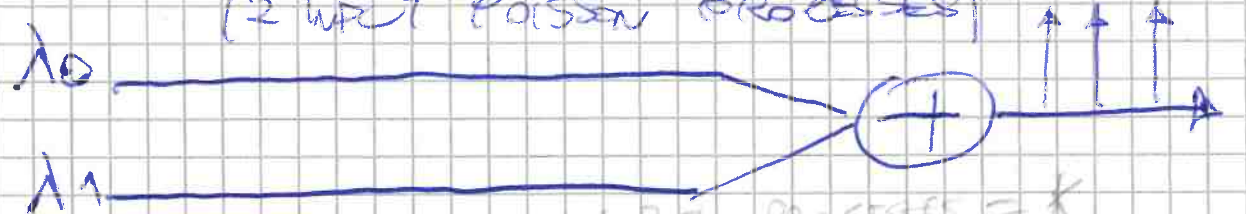
<u>OUTPUT</u>: A POISSON PROCESS

A Poisson Process (of parameter λ_i) has the following distribution:

$$P_{k_i}(t) = \frac{(\lambda_i t)^{k_i}}{k_i!} e^{-\lambda_i t} = P\{k_i(t) = k_i\}$$

$$P\{k(t) < k\} = \sum_{n=0}^{k} P\{k_i(t) = k\}$$

Now consider a 2-ARRIVAL PROCESS. $(n = 2)$
(2 INPUT POISSON PROCESSES)



CUSTOMERS ARRIVED = SUM of 2 INPUT PROCESSES = K

$$P\{K(t) = K\} = P\{K_1(t) + K_2(t) = K\}$$

$$n + (K - n) = K$$

Because of INDEPENDENCE

$$= \sum_{n=0}^{K} P\{K_1(t) = n, K_2(t) = K - n\}$$

$$\Rightarrow P\{K(t) = K\} = \underbrace{\sum_{n=0}^{K} P\{K_1(t) = n\} \cdot P\{K_2(t) = K - n\}}$$

$$= \sum_{n=0}^{K} \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} \frac{(\lambda_2 t)^{K-n}}{(K-n)!} e^{-\lambda_2 t}$$

$$= \frac{e^{-\lambda_1 t} e^{-\lambda_2 t}}{K!} \sum_{n=0}^{K} \frac{(\lambda_1 t)^n (\lambda_2 t)^{K-n} K!}{(K-n)!\, n!}$$

$$= \frac{e^{-\lambda_1 t} e^{-\lambda_2 t}}{K!} \underbrace{\sum_{n=0}^{K} \binom{K}{n} (\lambda_1 t)^n (\lambda_2 t)^{K-n}}$$

$$P\{K(t) = K\} = \frac{(e^{-\lambda_1 t} e^{-\lambda_2 t}) \left[(\lambda_1 + \lambda_2) t\right]^K}{K!}$$

$$= e^{-(\lambda_1+\lambda_2)t} \cdot \frac{[(\lambda_1+\lambda_2)t]^k}{k!}$$

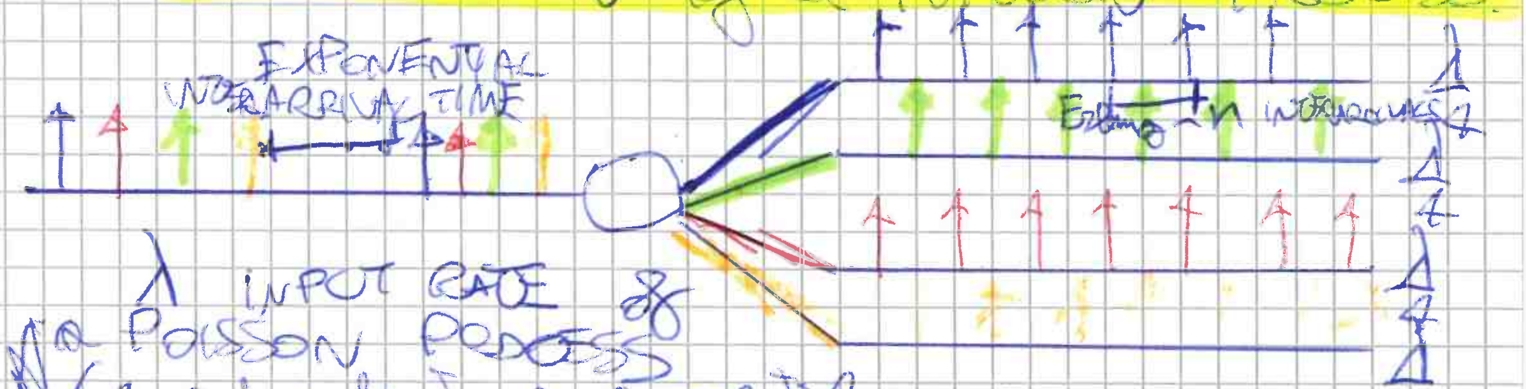
$$\boxed{\lambda_\Sigma = \lambda_1 + \lambda_2}$$

It is used in NETWORKS of queueing to state that, if we are given many INPUT POISSON PROCESSES, the SOURCE are still POISSONIAN

$$= e^{-\lambda_\Sigma t} \cdot \frac{[\lambda_\Sigma t]^k}{k!} = P\{K(t) = k\}$$

POISSON PROCESS (COMBINATION of n independent Poisson Processes)

36 DETERMINISTIC DECOMPOSITION of a POISSON PROCESS.

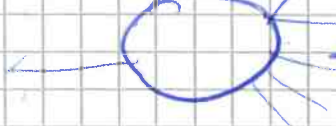


λ INPUT RATE of a POISSON PROCESS (4 independent & exponential RANDOM VARIABLES)

Erlang-4 INTERVAL

INPUT:

Exponentially identically distributed, independent RANDOM VARIABLES of a POISSON PROCESS.



= DETERMINISTIC DECOMPOSITION

"The "lane" to which a R.V. goes to is known a-priori.

OUTPUT:

No Poisson Process, but n Erlang-n INTERVALS

② $P_{ON}(t)$, $P_{OFF}(t)$:

Can be found by analyzing the STATE TRANSITION DIAGRAM for $N = 1$.

$$\Gamma_{off}: \quad \underset{P_{off}}{0} \xrightarrow{\lambda} \underset{P_{on}}{1}$$

$$\Gamma_{off}: \begin{cases} \dfrac{d}{dt} p_{off}(t) = -\lambda\, p_{off}(t) \\ P_{off}(0) = 1 \end{cases}$$

$\Rightarrow$ By $\mathcal{L}$:

$$s \cdot P_{off}(s) - p_{off}(0) = -\lambda\, p_{off}(s)$$

$$p_{off}(s)(s+\lambda) = 1$$

$$\Rightarrow p_{off}(s) = \frac{1}{s+\lambda}$$

$$\Rightarrow \boxed{p_{off}(t) = e^{-\lambda t} \cdot \mu(t)} = q$$

$$P_{ON}(t) + P_{OFF}(t) = 1$$

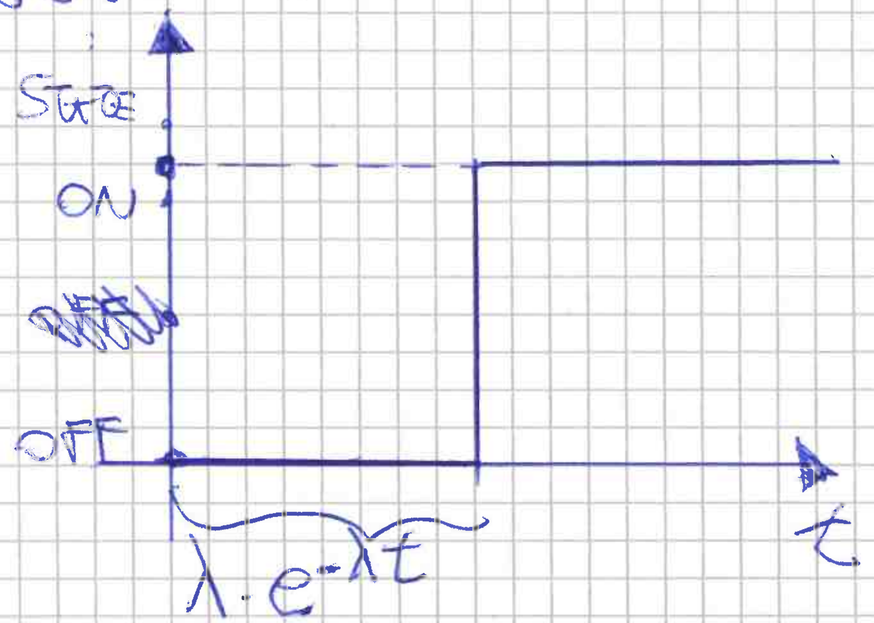
$$\Rightarrow P_{ON}(t) = 1 - P_{off}(t) =$$

$$\boxed{P_{on}(t) = \left(1 - e^{-\lambda t}\right)\mu(t)} = p$$

$$P_{ON}(t) = (1 - e^{-\lambda t})\,\mu(t) = p$$

$$P_{OFF}(t) = e^{-\lambda t}\,\mu(t) = q$$

$$\hookrightarrow f_{P_{OFF}}(t) = \lambda \cdot e^{-\lambda t}\,\mu(t) \Rightarrow \text{PDF of the } P_{OFF}(t)$$



For a TOTAL # ELEMENTS $= N$

\# ACTIVE ELEMENTS $= n$

BINOMIAL

$$P_n(t) = \binom{N}{n} p^n \cdot q^{N-n}$$

For $n \geq 0$:

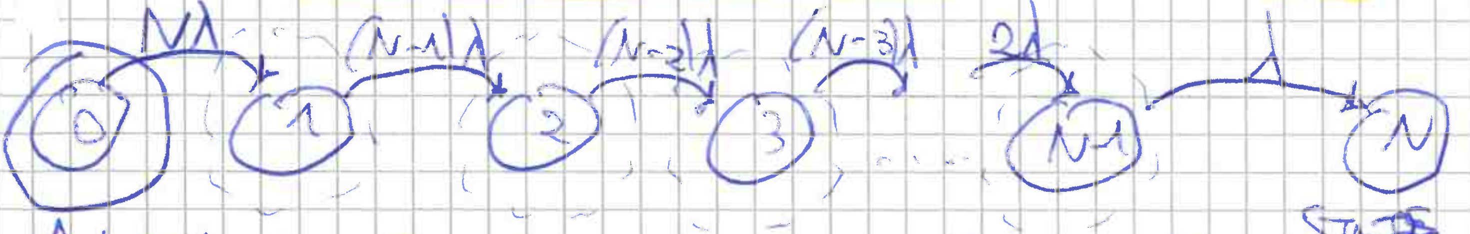
$$P_n(t) = \binom{N}{n} \cdot \underbrace{(1 - e^{-\lambda t})}_{P(ON)}{}^{n} \cdot \underbrace{(e^{-\lambda t})}_{P(OFF)}{}^{N-n}$$

$$E\{n(t)\} = N \cdot p = N \cdot (1 - e^{-\lambda t})\,\mu(t)$$

$$VAR\{n(t)\} = N \cdot pq = N \cdot (1 - e^{-\lambda t}) \cdot (e^{-\lambda t})$$

$\Rightarrow$ FORMAL PROOF $\Rightarrow$

(C) TRANSIENT-BEHAVIOUR ANALYSIS of CONTINUOUS-TIME BERNOULLI PROCESS.



N = MAX POPULATION

n = current # activated elements

For a generic STATE n:

FCP in TRANSIENT STATE:

$$\Gamma_0: \quad \begin{cases} \frac{d}{dt} p_0(t) = -N\lambda\, p_0(t) \\ \frac{d}{dt} p_1(t) = N\lambda\, p_0(t) - (N-1)\lambda\, p_1(t) \\ \frac{d}{dt} p_2(t) = (N-1)\lambda\, p_1(t) - (N-2)\lambda\, p_2(t) \\ \vdots \end{cases}$$

$$\Gamma_n: \quad \frac{d}{dt} p_n(t) = (N-n+1)\lambda\, p_{n-1}(t) - (N-n)\lambda\, p_n(t)$$

0) $\Rightarrow$ Apply $\mathcal{L}$ to Γ_0:

$$s \cdot P_0(s) - p_0(0) = -N\lambda\, P_0(s) \qquad p_0(0) = 1$$

$$P_0(s) \cdot (s + N\lambda) = 1$$

$$P_0(s) = \frac{1}{s + N\lambda}$$

1) $\Rightarrow$ Apply $\mathcal{L}$ to Γ_1:

$$s \cdot P_1(s) - \cancel{p_1(0)} = N\lambda\, P_0(s) - (N-1)\lambda\, P_1(s)$$

$$p_1(s)\cdot\left[s+(N-1)\lambda\right] = N\lambda\, p_0(s)$$

$$\Rightarrow p_1(s) = \frac{N\lambda\, p_0(s)}{\left[s+(N-1)\lambda\right]} = \frac{N\lambda}{(s+N\lambda)\cdot(s+(N-1)\lambda)}$$

2) Apply $\mathcal{L}$ to Γ_2:

$$s\cdot p_2(s) - \cancel{p_2(0)} = (N-1)\lambda\, p_1(s) - (N-2)\lambda\, p_2(s)$$

$$p_2(s)\cdot\left[s+(N-2)\lambda\right] = (N-1)\lambda\, p_1(s)$$

$$\Rightarrow p_2(s) = \frac{(N-1)\lambda\, p_1(s)}{\left[s+(N-2)\lambda\right]} = \frac{N\cdot(N-1)\lambda^2}{}$$

IN GENERAL, for Γ_n:

$$\boxed{p_n(s) = \frac{(N-n+1)\lambda}{s+(N-n)\lambda}\, p_{n-1}(s)}$$

$\Rightarrow$ Substituting $p_0(s), \dots, p_{n-1}(s)$ into $p_n(s)$:

$$p_n(s) = \underbrace{\overbrace{\frac{(N-n+1)\lambda}{s+(N-n)\lambda}}^{n=n}}_{p_n(s)} \cdot \underbrace{\overbrace{\frac{(N-n+2)\lambda}{s+(N-n+1)\lambda}}^{n=n-1}}_{p_{n-1}(s)} \cdots \underbrace{\overbrace{\frac{N\lambda}{s+(N-1)\lambda}}^{n=1}}_{p_1(s)} \cdot \underbrace{\overbrace{\frac{1}{s+N\lambda}}^{n=0}}_{p_0(s)}$$

LEFT-MOST

$0 \le n \le N$

$p_n(s)$ is the multiplication of $n+1$ poles (terms)

$$p_n(s) \xrightarrow{\mathcal{L}^{-1}} p_n(t) = \sum_{i=0}^{n} A_i\, e^{-(N-i)\lambda t}$$

$\downarrow$ DENOMINATOR

NB: Sum of all components is 1!

$\Rightarrow \binom{k}{k_1} r_1^{k_1} r_2^{k_2} = \frac{k!}{k_1! \cdot \underbrace{k_2!}_{k-k_1}} r_1^{k_1} \cdot r_2^{k_2}$

$\Rightarrow$ In general, for n:

$$\boxed{P\{k_1(t)=k_1, k_2(t)=k_2, \dots k_n(t)=k_n \mid K(t)=k\} = \frac{k!}{k_1! \cdot k_2! \cdot k_3! \cdot \dots \cdot k_n!} \cdot r_1^{k_1} \cdot r_2^{k_2} \dots r_n^{k_n}}$$

We now need to study the term: $P\{K(t)=k\}$ which we know is POISSONIAN

$P\{k(t)=k\} = \frac{(\lambda t)^k \cdot e^{-\lambda t}}{k!}$, $\mu(t) =$ $\sum_{i=1}^{n} r_i = 1$

$P\{k(t)=k\} = \frac{(\lambda t)^{k_1+k_2+\dots+k_n}}{k!} \cdot e^{-\lambda t \cdot \overbrace{(r_1+r_2+\dots+r_n)}^{1}}$

$\Rightarrow$ Now put everything together:

$P\{k_1(t)=k_1, k_2(t)=k_2, \dots, k_n(t)=k_n\}$

$= \cancel{k!} \cdot \frac{r_1^{k_1}}{k_1!} \cdot \frac{r_2^{k_2}}{k_2!} \cdot \dots \cdot \frac{r_n^{k_n}}{k_n!} \cdot \frac{(\lambda t)^{k_1} (\lambda t)^{k_2} \dots (\lambda t)^{k_n}}{\cancel{k!}}$

$\cdot e^{-\lambda r_1 t} \cdot e^{-\lambda r_2 t} \cdot \dots \cdot e^{-\lambda r_n t}$

$$= \underbrace{\frac{(r_1 \lambda t)^{k_1}}{k_1!} e^{-\lambda r_1 t}}_{\text{POISSON PROCESS with rate } r_1\lambda} \underbrace{\frac{(r_2 \lambda t)^{k_2}}{k_2!} e^{-\lambda r_2 t}}_{\text{POISSON PROCESS } (r_2\lambda)} \cdots \underbrace{\frac{(r_n \lambda t)^{k_n}}{k_n!} e^{-\lambda r_n t}}_{\text{POISSON PROCESS } (r_n\lambda)}$$

n - many POISSON PROCESSES!
as OUTPUT (independent from one another)

(3.8) CONTINUOUS-TIME BERNOULLI PROCESS

A CONTINUOUS-TIME BERNOULLI PROCESS has a set of N members that can be "switched on" in a time interval Δt (OFF ⇒ ON).

$\boxed{\text{STATE}}$ = # MEMBERS that are turned on (i.e. # activated elements)

INITIALLY: $n=0$ [All elements are SWITCHED OFF]

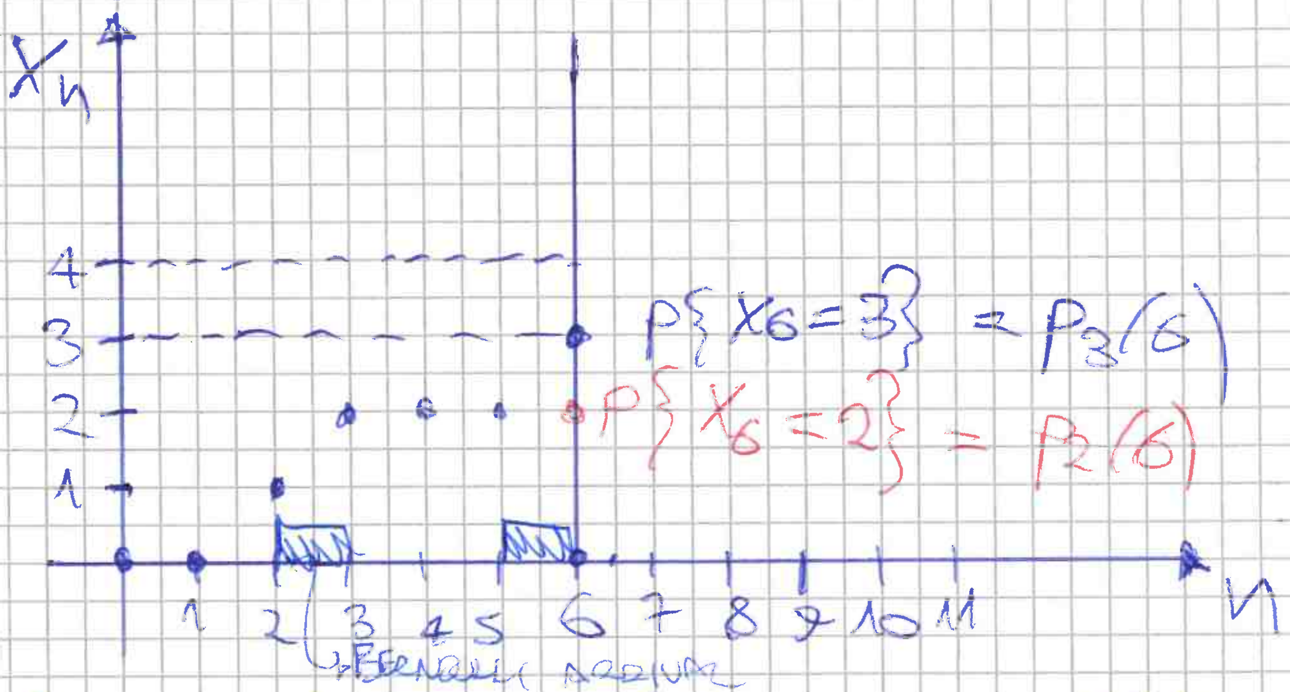
FINALLY: $n=N$ [All elements are SWITCHED ON]

USAGE / APPLICATION:

It is used in cellular / cable system [PBX ~~[illegible]~~] or 2G / 3G system
→ STATE of a CELL = # users using cells SYSTEM (i.e. # cells used)

Cooking
→ # POPCORNS that are Ready!

BINOMIAL DISTRIBUTION (VISUALIZED)



APPLICATION & USAGE of a BERNOULLI-PROCESS:

Used in systems with ARRIVALS to SLOTTED-TIME INTERVALS (Ex. PDH, SDH with ATM)

(2) For a Bernoulli process X:

$$G_X(z) = E\{z^X\} = \sum_{n=0}^{+\infty} p_n \cdot z^n \quad z \Leftrightarrow z^{-1}$$

$$E\{X\} = \frac{d}{dz} G_X(z)\Big|_{z=1} \quad \text{[MOMENT of ORDER 1]}$$

$$E\{X^2\} = \frac{d^2}{dz^2} G_X(z)\Big|_{z=1} + \frac{d}{dz} G_X(z)\Big|_{z=1} \quad \text{[MOMENT OF ORDER 2]}$$

~~We know $p_n = \binom{n}{}$~~

We know:

$$\boxed{P_n = \binom{N}{n} p^n \cdot q^{N-n}} \qquad \boxed{G_X(z) = \sum_{n=0}^{+\infty} P_n \cdot z^n}$$

$$\Rightarrow G_X(z) = \sum_{n=0}^{N} \binom{N}{n} p^n \cdot q^{N-n} \cdot z^n$$

$$= \sum_{n=0}^{N} \binom{N}{n} (p \cdot z)^n \cdot q^{N-n}$$

$$\cancel{E\{X\} \Rightarrow G_X(z) =}$$

We know:

$$\sum_{n=0}^{N} \binom{N}{n} a^{N-n} \cdot b^n = (a+b)^N$$

$$\Rightarrow \boxed{G_X(z) = (q+pz)^N}$$ X is a Bernoulli process

We can now compute $E\{X\}$, $E\{X^2\}$, $VAR\{X\}$

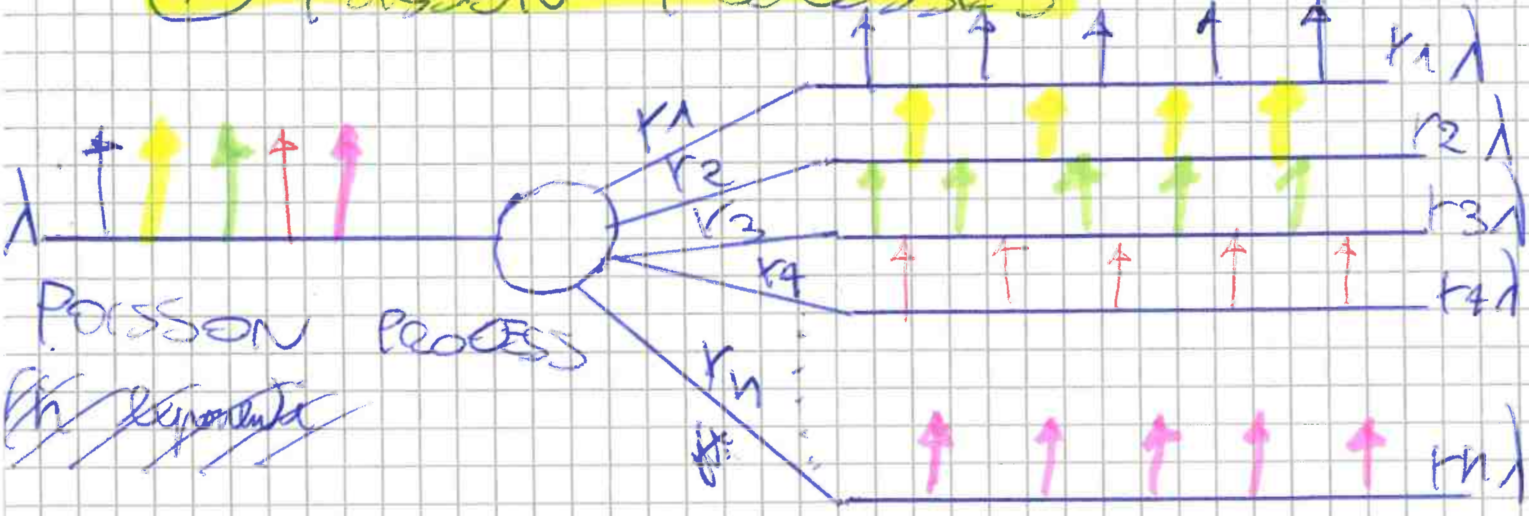
$$E\{X\} = N \cdot p \cdot (q+pz)^{N-1}\Big|_{z=1} = Np\underbrace{(p+q)}_{p+q=1}^{N-1}$$

Because:

$$\underbrace{E\{X=0\}}_{q} + \underbrace{E\{X=1\}}_{p} = 1$$

$$\Rightarrow \boxed{E\{X\} = Np}$$

37) STATISTICAL / PROBABILISTIC DECOMPOSITION of a POISSON PROCESS into (n) POISSON PROCESSES



$\boxed{r_i}$ = Probability to "pick" a certain LANE ⟹ (CONSERVATIVE DEVICE) ~~summing up to 1~~

$$\boxed{\sum_{i=1}^{n} r_i = 1}$$

↳ "Dice roll" upon deciding the lane.

INPUT:

One $\boxed{\text{POISSON PROCESS}}$ [Sum of n independent exponential R.V.s]

○ = Probabilistic choice of a lane.

OUTPUT:

n-independent $\boxed{\text{POISSON PROCESSES}}$ (one per lane)

PROOF: We need to study the joint PMF of the considered R.V.S.

~~KNOWS~~ ⟹ Show independence of the considered R.V.S. of the PROCESS.

$\boxed{K(t)}$ = # ARRIVALS (ALL LANES' ARRIVALS) in $[0,t]$ for the INPUT PROCESS

$\boxed{r_i}$ = P. of choosing the ith-lane

$K_i(t)$ = # ARRIVALS to lane i in $[0,t]$

$$\boxed{K(t) = \sum_{i=1}^{n} K_i(t)} = \text{Sum of all ARRIVALS}$$

$\Rightarrow$ Show independence of the DISCRETE RVS of the process.

$$P\{K_1(t)=k_1, K_2(t)=k_2, K_3(t)=k_3, \dots, K_n(t)=k_n\}$$

$$= P\{\underbrace{K_1(t)=k_1, K_2(t)=k_2, \dots, K_n(t)=k_n}_{A}, \underbrace{K(t)=k}_{B}\}$$

By the BAYES THEOREM: NOT ADDING ANYTHING NEW

$$P\{A, B\} = P\{A|B\} \cdot P\{B\}$$

POISSON-DISTRIBUTED $P\{K(t)=k\}$

$$= P\{K_1(t)=k_1, K_2(t)=k_2, \dots, K_n(t)=k_n \mid K(t)=k\} \cdot P\{K(t)=k\}$$

Now consider things for $n=2$ for sake of SIMPLICITY.

$$\boxed{P\{K_1(t)=\underset{p}{k_1}, K_2(t)=\underset{q}{k_2} \mid K(t)=k\}} \qquad k_2 = k-k_1$$

k_1 CHOICES of TYPE 1, k_2 CHOICES of TYPE 2 in k TRIALS [REMAINING ONES, WE k_1]

[~~BERNOULLI~~ BINOMIAL DISTRIBUTION]

$$P\{K_1(t)=k_1, K_2(t)=k_2 \mid K(t)=k\} = \binom{k}{k_1} r_1^{k_1} r_2^{k_2}$$

$$p_n(t) = \sum_{i=0}^{n} A_i \, e^{-(N-i)\lambda t} \qquad \mu(t)$$

by $\mathcal{L}^{-1}(\cdot) = +\ \mathcal{L}(\cdot) = (\cdot)$

$\to$ NUMERATOR of the fraction.

We now need to find A_i by PARTIAL FRACTION DECOMPOSITION

$$\boxed{A_i = P_n(s) \cdot [s + (N-i)\lambda]}\Bigg|_{s=-(N-i)\lambda}$$

$0 \le i \le n$

A_i has the following form

$$A_i = \frac{\overbrace{N \cdot (N-1) \cdot (N-2) \cdots (N-n+1)}^{\frac{N!}{(N-n)!}} \, \cancel{\lambda^n}}{\underbrace{[-(n-i)][-(n-i+1)] \cdots [-2][-1]}_{(n-i)!\,(-1)^{n-i}} \cdot \underbrace{[1][2] \cdots [i-1][i]}_{i!} \lambda}$$

For some terms, we have POSITIVE & NEGATIVE TERMS but they are all MULTIPLES of λ.

Recall: $\boxed{s = -(N-i)\lambda}$

For the LEFT-MOST term (POLE)

$$s + (N-n)\lambda \Big|_{s=-(N-i)\lambda}$$

$$= -\cancel{N\lambda} + i\lambda + \cancel{N\lambda} - n\lambda = i\lambda - n\lambda = \boxed{-(n-i)\lambda}$$

For the RIGHT-MOST TERM (POLE).

$$s + N\lambda \Big|_{s=-(N-i)\lambda} = -\cancel{N\lambda} + i\lambda + \cancel{N\lambda} = \boxed{i\lambda}$$

We now now substitute A_i into $p_n(t)$ (after some modifications)

$$A_i = \frac{N!}{(N-n)!\,(n-i)!\,(-1)^{n-i}\,i!}$$

$$A_i = \frac{N!}{(N-n)!\,(n-i)!\,(-1)^{n-i}\,i!}$$

$\Rightarrow$ Substitute A_i into $P_n(t)$

$$P_n(t) = \sum_{i=0}^{n} A_i \cdot e^{-(N-i)\lambda t} \cdot \mu(t)$$

TAKES (-1) [illegible]

$$= \sum_{i=0}^{n} \frac{N!\,(-1)^{n-i}}{(N-n)!\,(n-i)!\,i!} \cdot e^{-(N-i)\lambda t}$$

$n-i$

NEW

$$= \underline{e^{-N\lambda t}}\, e^{n\lambda t} \sum_{i=0}^{n} \frac{N!\,n!\,(\lambda)^i}{(N-n)!\,n!\,(n-i)!\,i!} (e^{-\lambda t})$$

$= \binom{N}{n}$

$$= e^{-N\lambda t} \cdot e^{n\lambda t} \binom{N}{n} \sum_{i=0}^{n} \frac{n!\,(1)^i}{(n-i)!\,i!} (e^{-\lambda t})^{n-i}$$

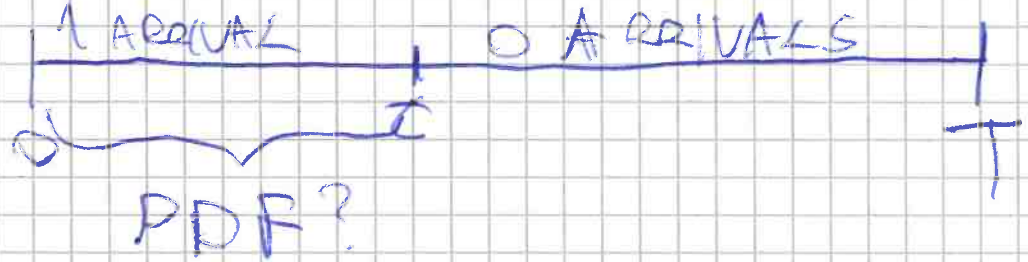
$$= e^{-N\lambda t} \cdot e^{n\lambda t} \binom{N}{n} \sum_{i=0}^{n} \frac{n!}{(n-i)!\,i!} (-e^{-\lambda t})^{n-i} \cdot (1)$$

$$= \binom{N}{n} \underbrace{e^{-(N-n)\lambda t}}_{q^{N-n}} \underbrace{(1-e^{-\lambda t})^n}_{p^n}$$

$\Rightarrow$ BINOMIAL DISTRIBUTION of CONTINUOUS-TIME BERNOULLI PROCESS

$$e^{-N\lambda t} \cdot e^{n\lambda t} \binom{N}{n} \underbrace{\sum_{i=0}^{n} \binom{n}{i} (-e^{-\lambda t})^{n-i} (1)^i}_{(1-e^{-\lambda t})^n}$$

(39) ~~Poisson~~ UNIFORM PDF of the INSTANT of ARRIVAL over an interval (0,t] for POISSONIAN ARRIVALS



What is the PDF of POISSONIAN arrivals in [0,t]?

⇒ We want to draw it is UNIFORM-DISTRIB. ~~POISSONIAN~~

$$P\{1 \text{ arrival in } (0,t] \mid 1 \text{ arrival in } (0,T]\}$$

By the BAYES THEOREM:

$$P\{A \mid B\} = \frac{P\{A, B\}}{P\{B\}}$$

$$P\{\overset{A}{1 \text{ ARR in } (0,t]} \mid \overset{B}{1 \text{ ARR in } (0,T]}\} = \cancel{\frac{P\{1 \text{ ARR in } (0,t] \mid (0,T]\}}{P\{1 \text{ ARR in } (0,T]\}}}$$

$$= \frac{P\{1 \text{ arrival in } (0,t],\ 1 \text{ arrival in } (0,T]\}}{P\{1 \text{ arrival in } (0,T]\}}$$

$$= \frac{P\{1 \text{ arrival in } (0,t],\ 0 \text{ arrivals in } (t,T]\}}{P\{1 \text{ arrival in } [0,T]\}}$$

We know: POISSON DISTRIBUTION [K arrivals]

$$P_K(t) = \frac{(\lambda t)^K}{K!} \cdot e^{-\lambda t} \cdot u(t)$$

$$P\{K \text{ arrivals in } (0,t]\} = \frac{(\lambda t)^K}{K!} e^{-\lambda t} \cdot \mu(t)$$

$$P\{1 \text{ arrival in } (0,t]\} = (\lambda t)\, e^{-\lambda t} \cdot \mu(t)$$

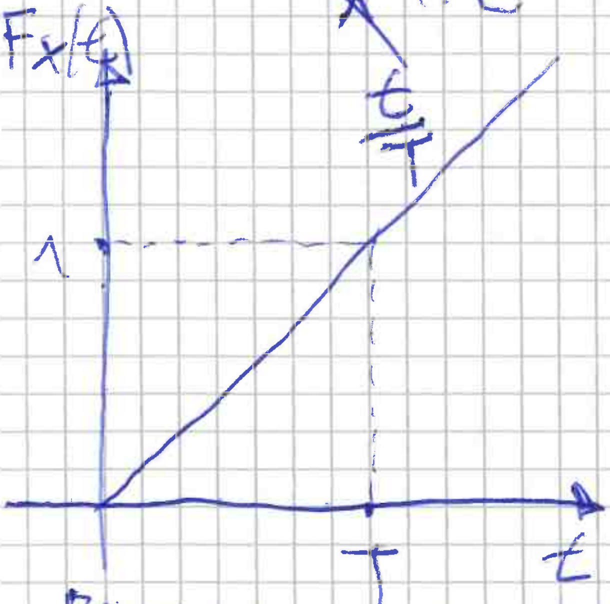
$$P\{0 \text{ arrivals in } (t,T]\} = \frac{\cancel{(\lambda t)^0}}{\cancel{0!}} e^{-\lambda(T-t)} \cdot \mu(t)$$

$$P\{1 \text{ arrival in } (0,T]\} = \lambda T \cdot e^{-\lambda T} \cdot \mu(T)$$

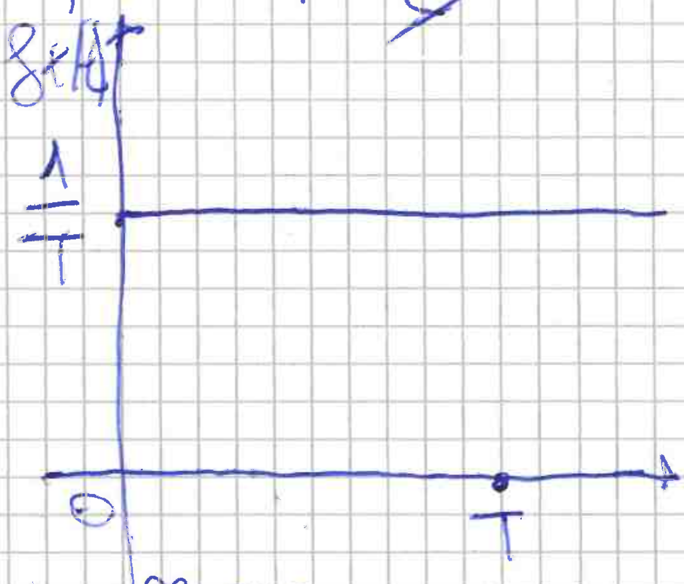
=> Because the POISSON PROCESS is HOMOGENEOUS & EVENTS in DISJOINT intervals are INDEPENDENT:

$$\frac{P\{1 \text{ arrival in } (0,t],\ 0 \text{ arrivals in } (t,T]\}}{P\{1 \text{ arrival in } (0,T]\}} = \boxed{\frac{t}{T}}$$

$$= \frac{(\cancel{\lambda} t)\, \cancel{e^{-\lambda t}} \cdot \cancel{e^{-\lambda(T-t)}}}{\cancel{\lambda} T \cdot e^{-\lambda T} \cdot \mu(T)} = \frac{t}{T} \cdot \frac{\cancel{e^{-\lambda T}}}{\cancel{e^{-\lambda T}}} = \boxed{\frac{t}{T}}$$



PROBABILITY DISTRIBUTION FUNCTION $F_X(t)$



PROBABILITY DENSITY FUNCTION $f_X(t)$

⇒ If ARRIVALS are POISSONIAN - DISTRIBUTED in $(0,T)$ ⇒ They are JOINTLY - UNIFORMLY DISTRIBUTED on $(0,T)$

If arrivals are UNIFORMLY - DISTRIBUTED on an interval $(0,T)$ $\not\Rightarrow$ ~~JOINT~~ POISSONIAN ARRIVALS in $(0,T)$

40 KENDALL'S NOTATION:

A | B | C | D | E - X

- A → ARRIVAL'S PROCESS
- B → SERVICE' PROCESS
- C → # SERVERS N_S (Integer)
- D → N_W = Capacity of waiting line
- E → QUEUEING DISCIPLINE (FCFS | LIFO RR, PS)
- X → N_U = # CUSTOMERS in the population

M = EXP. DISTRIBUTED (MARKOVIAN)

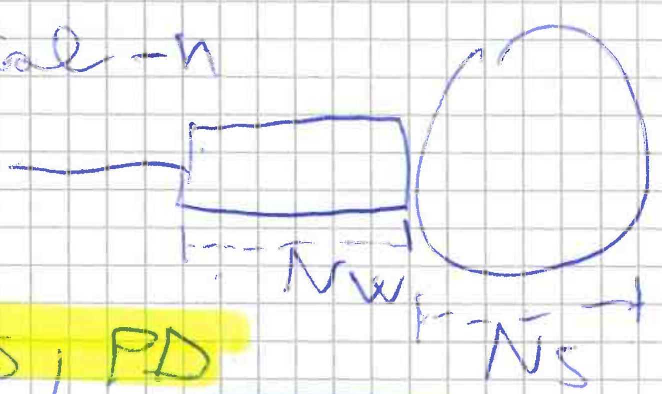
GEO = GEOMETRICALLY DISTRIBUTED

D = Deterministic

Hn = hyperexponential - n

En = Erlang - n

G = General



41 PB, PL, PBS, PD

$$\boxed{P_B = P\{n = N\}}$$

(Servers & waiting line both busy)

$N = N_w + N_s$

$$P_L = P\left\{ \text{Blocked queue} \,\middle|\, \begin{matrix}\text{Request of service} \\ \text{offered in } dt\end{matrix} \right\}$$

by the Bayesian Theorem:

$$P\{A \mid B\} = \frac{P\{A, B\}}{P\{B\}}$$

$$P_L = \frac{P\{\overset{B}{\text{Blocked queue}}, \overset{A}{\text{Request of service offered in } dt}\}}{P\{\text{Request of service offered in } dt\}}$$

$$P\{A, B\} = P\{A \mid B\} \cdot P\{B\}$$

$$\underbrace{\cancel{P\{A, B\}}}_{P_L} = \frac{P\{\text{Request of service offered in } dt \mid \text{Blocked queue}\} \cdot P_B}{\underbrace{P\{\text{Request of service offered in } dt\}}_{\sum_{n=0} P_n \cdot \lambda_n \quad \lambda_n = \lambda}}$$

For a Poisson process:

$$P_L = P_B \qquad \cancel{\frac{\lambda\, dt}{\lambda\, dt}}$$

[i.e. fact that queue is blocked has no IMPACT]

$$\Rightarrow \boxed{P_L = P_B}$$

If $P_B > 0$ ~~$\Rightarrow$~~ $P_L > 0$

$$P_L = \frac{P\{\text{Request of service rejected in } dt\} - \#\text{REJ}}{P\{\text{Request of service offered in } dt\} - \#\text{OFF}}$$

~~P_B~~

$$\boxed{P_{BS} = P\{n \geq N_S\}}$$

in QUEUE (WAITING LINE)

$$P_D = P\{\text{Blocked service} \mid \text{Request of service accepted in } dt\}$$

By the Bayes Theorem:

$$P\{A|B\} = \frac{P\{A, B\}}{P\{B\}} \qquad P\{A, B\} = P\{A|B\} \cdot P\{B\}$$

ACCEPTED with delay

$$P_D = \frac{P\{\text{Request of service accepted in } dt \mid \text{Blocked service}\}}{P\{\text{request of service accepted in } dt\}}$$

ACCEPTED

$$P_D = \frac{P_{BS} \cdot P\{\text{Request of service accepted in } dt \mid \text{Blocked service}\}}{P\{\text{Request of service accepted in } dt\}}$$

For a POISSON PROCESS

$$P_D = P_{BS} \cdot \frac{\cancel{\lambda dt}}{\cancel{\lambda dt}}$$

$$\Rightarrow P_D = P_{BS}$$

NB:

Blocked QUEUE $\neq$ Blocked SERVICE

② PB, PL, PBS, PD for a MARKOVIAN QUEUE

$$P_B = P_N = P\{n = N\}$$

(Only in case of FINITE WAITING LINE)

$$P_B = P_L = \frac{\#\text{REFUSED} \; P\{\text{Request of service offered in } dt \mid \text{Blocked queue}\}}{\#\text{OFFERED} \; P\{\text{Request of service offered in } dt\}}$$

$$= \frac{\lambda_N \cancel{dt} \, P_B}{\sum_{i=0}^{N} p_i \, \lambda_i \cancel{dt}} = P_B \frac{\cancel{\lambda}}{\cancel{\lambda} \sum_{i=0}^{N} p_i} \qquad \sum_{i=0}^{N} p_i = 1$$

$$\Rightarrow \boxed{P_L = P_B}$$

$$\boxed{P_{BS} = P\{n \geq N_S\} = \sum_{n=N_S}^{N} P_n} \boxed{= P_D}$$

$$P_D = P\{\text{Request of service accepted } \cancel{\text{with}} \text{ in } dt \mid \text{Blocked service}\}$$

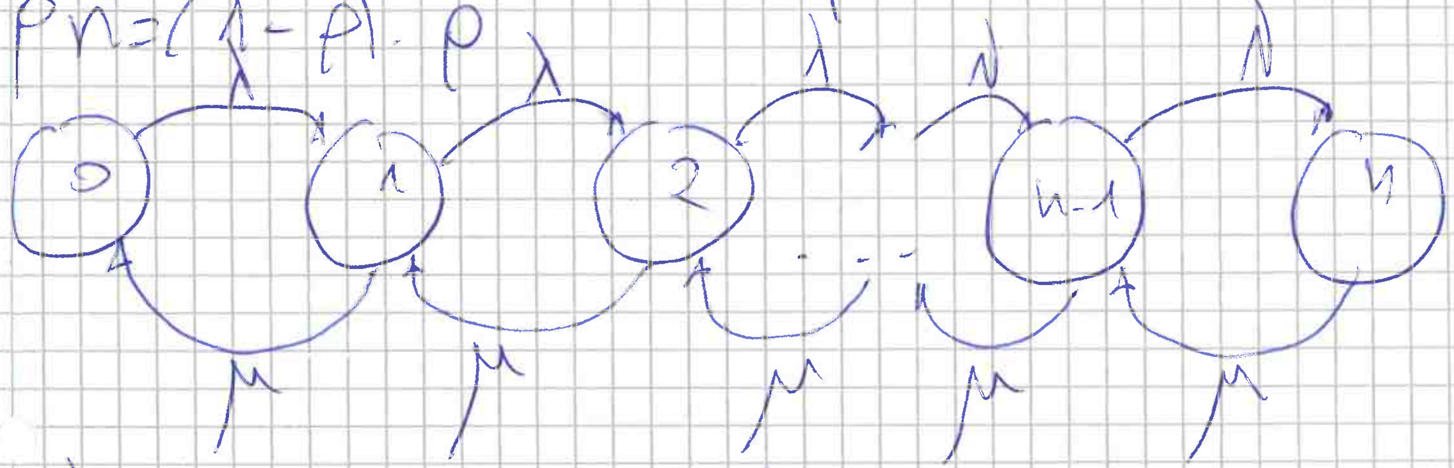
FOR POISSON INPUT ($N \to \infty$): $P\{\text{Request of service accepted in } dt\}$

$$P_D = P_{BS} \frac{\cancel{\lambda dt}}{\cancel{\lambda dt}} \qquad = \cancel{\frac{\sum_{k=N_S}^{N-1} \lambda_k p_k \, dt}{\sum_{k=0}^{N} \lambda_k p_k \, dt}}$$

$$=$$

(42) (b) P_B, P_L, P_{BS}, P_D, P_N, $E\{n\}$, $E\{n_w\}$, $E\{n_s\}$ for M|M|1 QUEUE

$$p_n = (1-\rho)\cdot\rho^n$$



$$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0$$

$$\lambda p_1 = \mu p_2 \Rightarrow p_2 = \frac{\lambda}{\mu} p_1 = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

$$\lambda p_2 = \mu p_3 \Rightarrow p_3 = \frac{\lambda}{\mu} p_2 = \left(\frac{\lambda}{\mu}\right)^3 p_0$$

$$\Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 \qquad \frac{\lambda}{\mu} = \rho$$

$$\Rightarrow p_n = (\rho)^n p_0$$

$$\sum_{n=0}^{+\infty} p_n = 1 \Rightarrow \sum_{n=0}^{+\infty} p_0 \cdot \rho^n = 1$$

$$\Rightarrow p_0 \triangleq \frac{1}{\sum_{n=0}^{+\infty} \rho^n}$$

$$\Rightarrow p_0 = \frac{1}{\frac{1}{1-\rho}} = 1-\rho$$

$\Rightarrow$ Indeed: $p_n = (1-\rho)\cdot\rho^n$

$$\boxed{P_n} = (1-\rho)\cdot\rho^n \Rightarrow E\{n\} = \sum_{i=0}^{+\infty} n\, p_n$$

$$\boxed{E\{n\}} = \sum_{n=0}^{+\infty} n(1-\rho)\cdot\rho^n = (1-\rho)\sum_{n=0}^{+\infty} n\cdot\rho^n$$

$$= \cancel{(1-\rho)}\,\frac{\rho}{(1-\rho)^{\cancel{2}}} = \frac{\rho}{1-\rho} = E\{n\}$$

$$\boxed{\rho} = \frac{E\{ns\}}{Ns} = \frac{\rho}{1} = \frac{\lambda}{\mu} = A$$

$$\boxed{E\{ns\}} = 0\cdot p_0 + 1 p_1 = 1(1-p_0) = \cancel{1-}(1-\rho)$$

$$E\{ns\} = \rho = \frac{\lambda}{\mu}$$

$$E\{ns\} = \sum_{n=0}^{Ns} n\cdot p_n$$

$$\boxed{\boxed{E\{nw\}} = \sum_{i=1+Ns}^{+\infty} (i-Ns)\,p_i} = \sum_{i=1}^{+\infty} (i-1)\,p_i$$

$$= \sum_{i=1}^{+\infty} (i-1)\,p_i = \sum_{i=1}^{+\infty} i\cdot p_i - \sum_{i=1}^{+\infty} p_i$$

$$= \sum_{i=1}^{+\infty} i\cdot(1-\rho)\cdot\rho^i - \sum_{i=1}^{+\infty}(1-\rho)\cdot\rho^i$$

$$= (1-\rho)\cdot\sum_{i=1}^{+\infty} i\cdot\rho^i - (1-\rho)\cdot\rho\sum_{i=1}^{+\infty}\rho^{i-1}$$

$$= \cancel{(1-\rho)}\,\frac{\rho}{(1-\rho)^{\cancel{2}}} - \frac{\cancel{(1-\rho)}\,\rho\cdot 1}{\cancel{(1-\rho)}}$$

$$\Rightarrow E\{n_w\} = \frac{\rho}{(1-\rho)} - \rho = \frac{\rho-(1-\rho)\cdot\rho}{(1-\rho)} = \frac{\cancel{\rho}-\cancel{\rho}+\rho^2}{(1-\rho)}$$

$$\Rightarrow E\{n_w\} = \frac{\rho^2}{(1-\rho)}$$

$\boxed{P_B} = P\{n=N\} = 0$ because $N_w = \infty$

$\Rightarrow P_D = P_B = 0$

$$\boxed{P_{BS}} = P\{n \geq N_S\} = \sum_{i=N_S}^{+\infty} p_i = \overset{=\rho^1}{\sum_{i=1}^{+\infty} p_i} = 1 - p_0$$

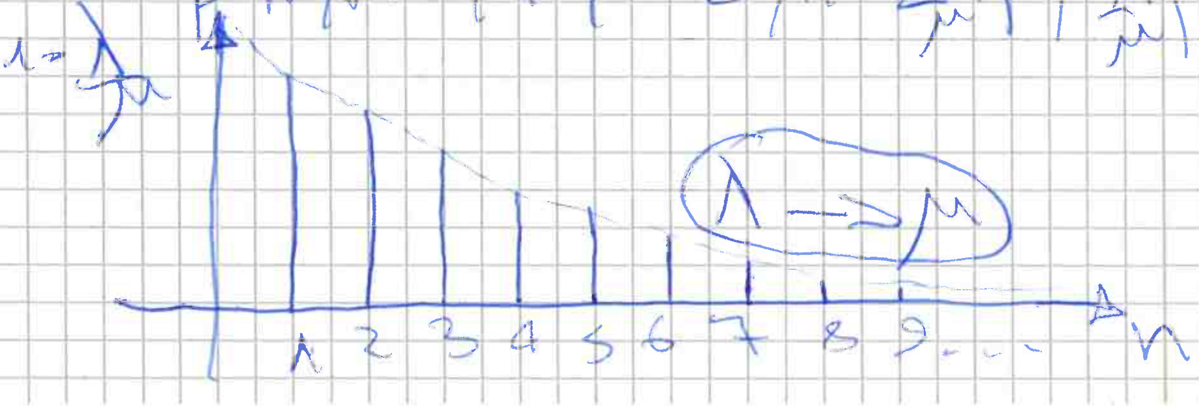
$$= \cancel{1} - \cancel{1} + \rho = \rho = P_D$$

$$\boxed{E\{n\}} = E\{T\} \cdot \Lambda \Rightarrow E\{T\} = \frac{E\{n\}}{\Lambda}$$

$$\Rightarrow \boxed{E\{T\}} = \frac{\overset{E\{n\}}{\rho}}{\Lambda(1-\rho)} = \frac{\frac{\Lambda}{\mu}}{\Lambda\cdot(1-\rho)}$$

STEADY-STATE PROBABILITY PLOT

$$P_n = (1-\rho)\cdot\rho^n = \left(1-\frac{\Lambda}{\mu}\right)\cdot\left(\frac{\Lambda}{\mu}\right)^n$$



$$= \frac{\sum_{k=N_S}^{N-1} \lambda_k p_k \, dt}{\sum_{k=0}^{N-1} \lambda_i p_i \, dt} = \frac{P\{\text{Request of service accepted with delay in } dt\}}{P\{\text{Request of service accepted in } dt\}}$$

42

$E\{T\}$, $E\{T_S\}$, $E\{T_W\}$ $\qquad E\{T\} = E\{T_S\} + E\{T_W\}$

$E\{n\}$ $\quad E\{n_S\}$ $\quad E\{n_W\}$

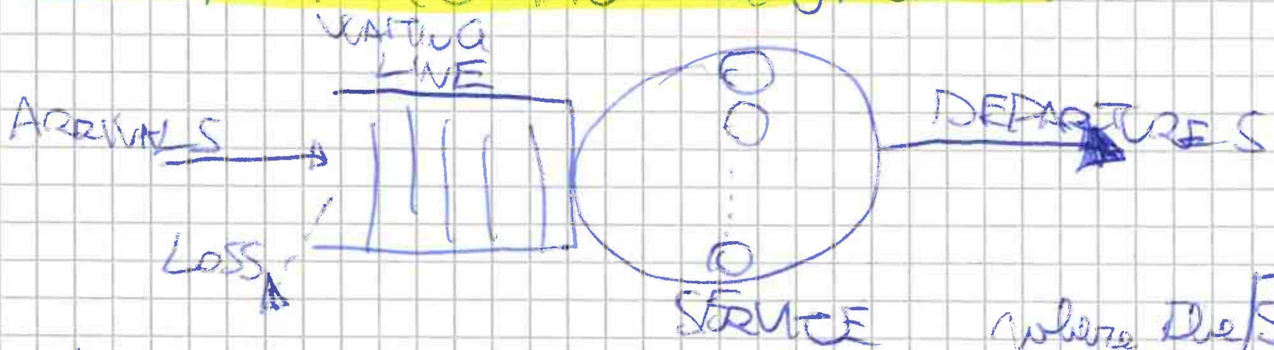
$\boxed{E\{n\}} = \Lambda \cdot E\{T\} \Rightarrow E\{T\} = \frac{E\{n\}}{\Lambda} \qquad \boxed{E\{n\} = \sum_{n=0}^{\infty} n \, p_n}$

$\boxed{E\{n_S\}} = \Lambda \cdot E\{T_S\} \Rightarrow E\{T_S\} = \frac{E\{n_S\}}{\Lambda} \qquad E\{n_S\} = \sum_{n=0}^{N_S} n \, p_n$

$E\{n_W\} = \Lambda \cdot E\{T_W\} \Rightarrow E\{T_W\} = \frac{E\{n_W\}}{\Lambda}$

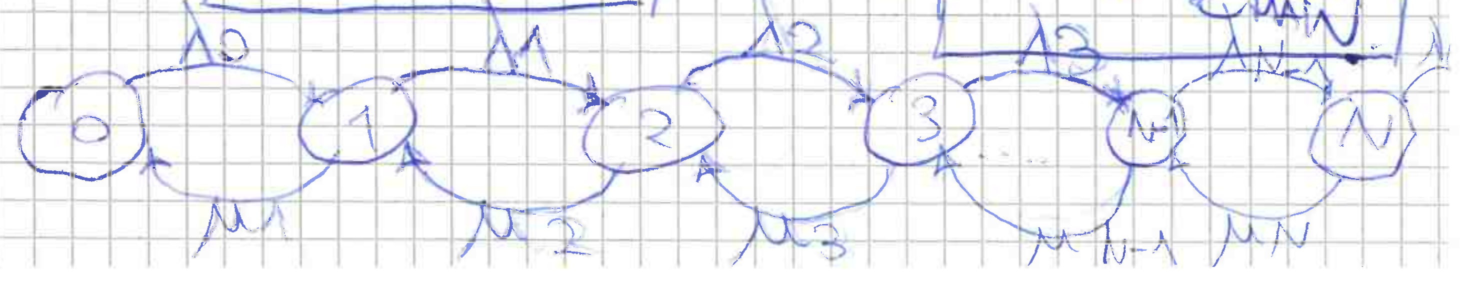
$E\{n\} = E\{n_S\} + E\{n_W\}$

43 MARKOVIAN QUEUES



a) A Markovian Queue is a QUEUE where the STATE is characterized by the # CUSTOMERS (and only by that!)

b) The evolution of the STATE of the system is a MARKOV CHAIN and a B & D MARKOV CHAIN



(44) **A = TRAFFIC INTENSITY**

$$\boxed{A = \Lambda \cdot E\{T_S\}} = \frac{\lambda}{\mu} \quad \text{for } M/M/1$$

(a) If $E\{n_W\} = 0$

$$\Rightarrow E\{n\} = \underbrace{E\{n_S\}}_{A} = A$$

$$\rho = \frac{A}{N_S} < 1 \Rightarrow A < N_S$$

(b) ERGODICITY CONDITION:

$$\boxed{A < N_S} \Rightarrow A < 1 \quad \left(\text{For } M/M/1 \text{ QUEUE}\right)$$

(45) **TRAVERSAL TIME $E\{T\}$ in M/M/1 QUEUE, $E\{T_W\}$ by PASTA PROPERTY**

$$E\{T\} = E\{T_W\} + E\{T_S\}$$

$$\boxed{E\{T\} = \int_0^{+\infty} x \cdot f_X(x)\, dx}$$

$$\boxed{E\{T_S\} = \frac{1}{\mu}} \quad \boxed{E\{T_W\}} \text{ by the PASTA PROPERTY}$$

$$\boxed{E\{T_W\} = \sum_{i=0}^{+\infty} E\{T_W | i\}\, p_i^{[arr]}}$$

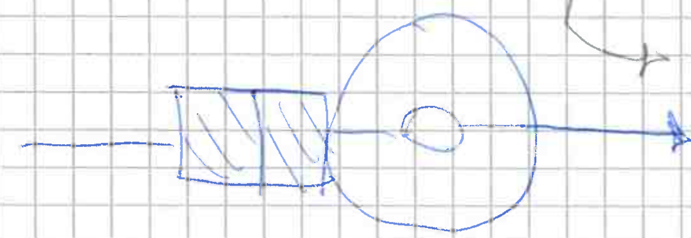
IF 0 CUSTOMERS in queue: $E\{T_W | 0\} = 0$

$\Rightarrow$ No time spent in the waiting line.

1) There ~~are~~ is 1 customer in the queue.

$E\{T_w|1\} = \frac{1}{\mu}$ [Waiting for 1 customer to be served]

$E\{T_w|2\} = \frac{2}{\mu}$ [Waiting for 2 customers to be served]



$\frac{1}{\mu} + \frac{1}{\mu} = \frac{2}{\mu}$

PASTA PROPERTY

"Poisson Arrivals See Time Averages"

$$\boxed{E\{T_w\} = \sum_{i=0}^{+\infty} E\{T_w|i\} \cdot p_i^{[ARR]}},$$

$$p_i^{(arr)} = p_i \to P\{X_n = i\}$$

$$E\{T_w\} = \sum_{i=1}^{+\infty} \underbrace{i \frac{1}{\mu}}_{E\{T_w|i\}} p_i = \frac{1}{\mu} \sum_{i=1}^{+\infty} i \cdot p_i$$

"$E\{n\}$"

"$\frac{\rho}{1-\rho}$"

$$\Rightarrow \boxed{E\{T_w\} = \frac{1}{\mu} \cdot \frac{\rho}{(1-\rho)}}$$

$\rho \to 0 \Rightarrow E\{T_w\} = 0$ [0 UTILIZATION ⇓ WAITING TIME is 0]

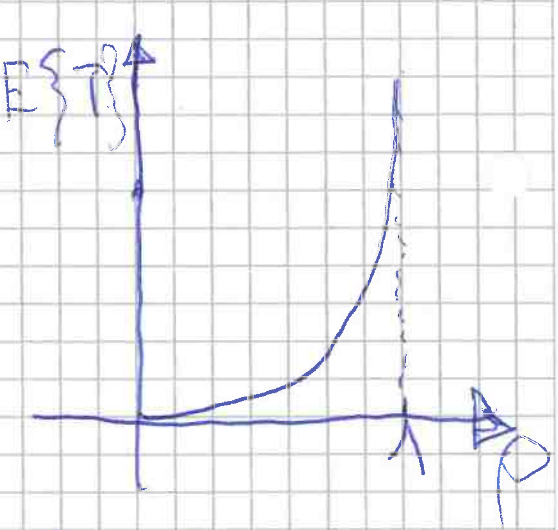
$\rho \to 1 \Rightarrow E\{T_w\} = \infty$ 1 UTILIZATION ⇓ WAITING TIME is ∞

$$\Rightarrow E\{T\} = E\{T_w\} + E\{T_S\}$$

$$= \frac{1}{\mu}\left[\frac{\rho}{1-\rho} + 1\right]$$

$$= \frac{1}{\mu}\left[\frac{\rho + 1 - \rho}{(1-\rho)}\right] = \frac{1}{\mu}\,\frac{1}{(1-\rho)}$$

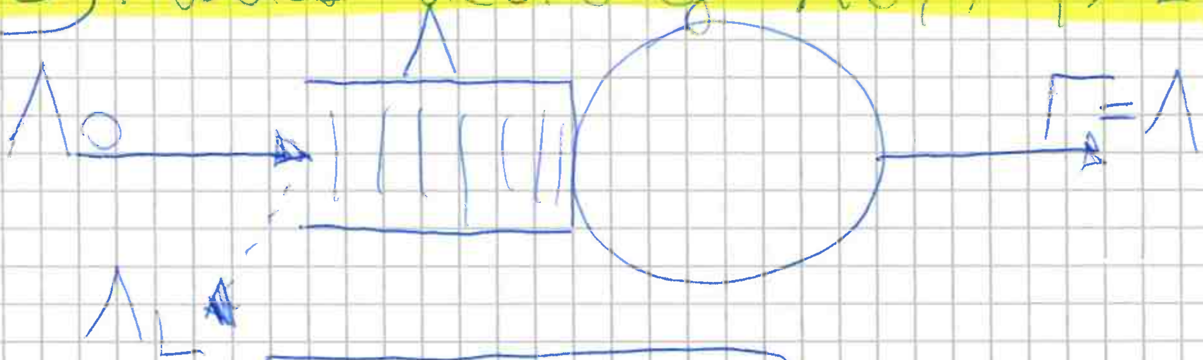
$$\Rightarrow E\{T\} = \frac{\frac{1}{\mu}}{1-\rho}$$



$\rho \to 0 \Rightarrow E\{T\} = \frac{1}{\mu}$

$\rho \to 1 \Rightarrow E\{T\} = \infty$

(46) AVERAGE VALUES of $\Lambda_0, \Lambda, \Lambda_L, \Gamma, \Gamma_{MAX}$



$$\boxed{\Lambda_0 = \Lambda + \Lambda_L}$$

$$\boxed{\Gamma = \Lambda = \Lambda_0 (1 - P_L)}$$

$$\boxed{\Lambda_L = \Lambda_0 \cdot P_L}$$

$\Lambda < \Gamma_{MAX}$ with Ns servers:

$$\Gamma_{MAX} = \mu \cdot N_S = \frac{N_S}{E\{T_S\}}$$

$$\Lambda < \frac{N_S}{E\{T_S\}} \qquad \underbrace{\Lambda E\{T_S\}}_{} < N_S$$

For a MARKOVIAN QUEUE: A = TRAFFIC INTENSITY

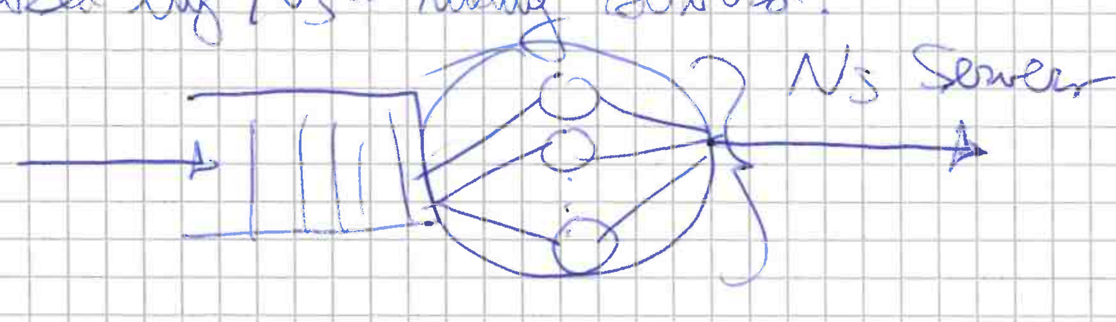
$$\boxed{\Lambda_O = \sum_{k=0}^{N} \lambda_k p_k}$$ (everything)

$$\boxed{\Lambda_L = \lambda_N p_N = \lambda_N \cdot P_B}$$ (just saturated)

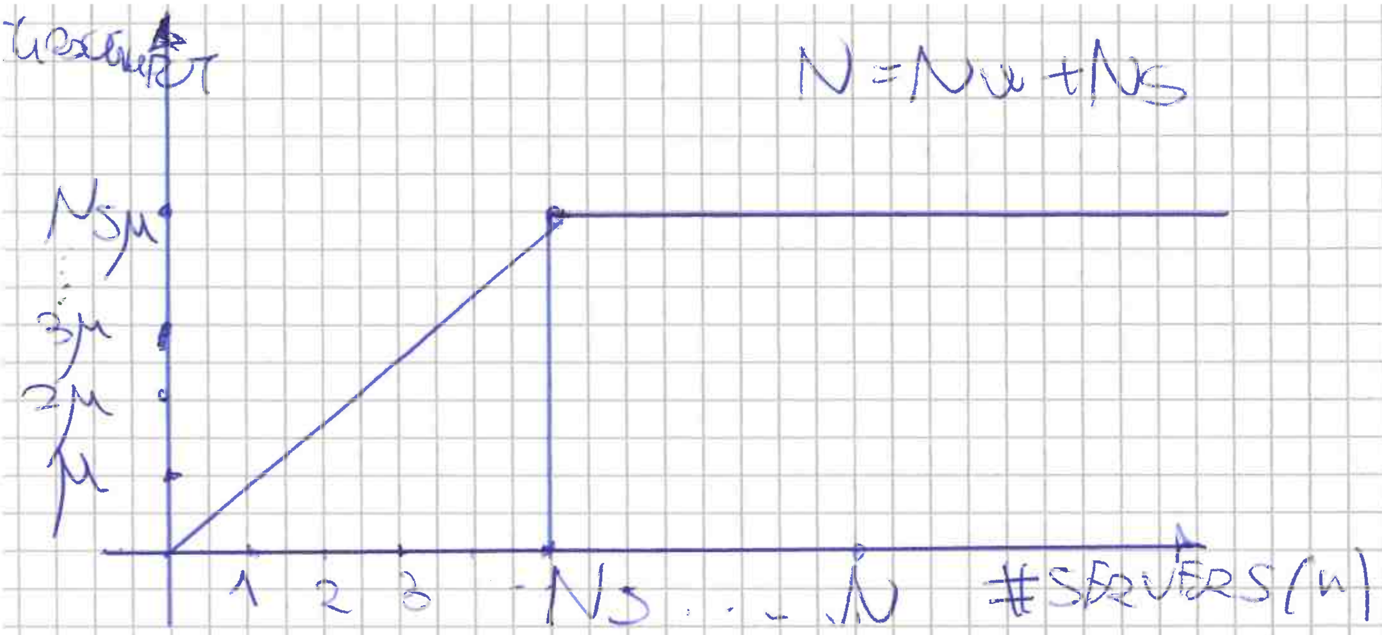
$$\boxed{\Lambda = \Lambda_O - \Lambda_L = \sum_{k=0}^{N-1} \lambda_k p_k = (1-P_B)\Lambda_O}$$ (excluding saturated)

47) M/M/Ns QUEUES STATE PROBABILITY OCCUPANCY p_n

An M/M/Ns QUEUE is a queue characterized by Ns-many servers.



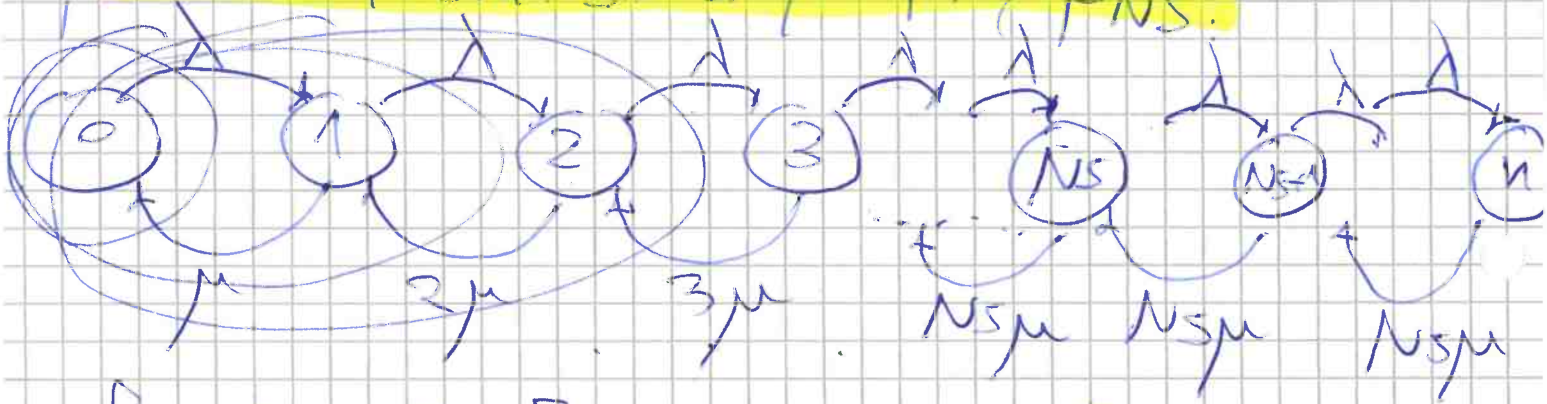
Each server has exponential service & is independent from the other ones.
[MEMORYLESS & MARKOVIAN property hold]



$N = N_W + N_S$

a) We increase the THROUGHPUT up to $N_S \cdot \mu$ as the # SERVERS increases: Every server can process customers with rate μ ⇒ So, increasing the # SERVERS, we increase the # customers processable at the same time up to N_S (max # SERVERS in the queue)

b) STATE PROBABILITY P_n, P_{N_S}.



By the FCP in STEADY-STATE

Γ_1: $\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0$

Γ_2: $\lambda p_1 = 2\mu p_2 \Rightarrow p_2 = \frac{\lambda}{2\mu} p_1 = \left(\frac{\lambda}{\mu}\right)^2 \frac{p_0}{2}$

Γ_3: $\lambda p_2 = 3\mu p_3 \Rightarrow p_3 = \frac{\lambda}{3\mu} p_2 = \left(\frac{\lambda}{\mu}\right)^3 \frac{p_0}{3 \cdot 2}$

Γ_n: $\Rightarrow \boxed{p_n = \left(\frac{\lambda}{\mu}\right)^n \frac{p_0}{n!}}$ $\quad 0 \leq n \leq N_S$

$$\boxed{P_{NS} = \left(\frac{\lambda}{\mu}\right)^{NS} \cdot \frac{1}{NS!} \cdot P_0}$$

Γ_{NS}: $\lambda P_{NS} = NS\mu \, P_{NS+1} \Rightarrow \boxed{P_{NS+1} = \frac{\lambda}{NS\mu} P_{NS}}$

Γ_{NS+1} $\lambda P_{NS+1} = NS\mu \, P_{NS+2} \Rightarrow \boxed{P_{NS+2} = \frac{\lambda}{NS\mu} P_{NS}}$

$$\Rightarrow P_{NS+2} = \frac{\lambda}{NS\mu} \cdot \frac{\lambda}{NS\mu} P_{NS} = \left(\frac{\lambda}{NS\mu}\right)^2 P_{NS}$$

$$\Rightarrow P_{NS+i} = \left(\frac{\lambda}{\mu}\right)^{NS+i} \cdot \frac{1}{NS!\, NS^{i}}$$

$$\Rightarrow P_{NS+i} = \left(\frac{\lambda}{NS\mu}\right)^{i} P_{NS} \qquad n \geq NS, \; i \geq 0$$

Throughput cannot be (more than the max. # servers in the queue).

(c) ERGODICITY CONDITION:

We have found in (a) that:

$$\boxed{P_n = \left(\frac{\lambda}{\mu}\right)^{n} \cdot \frac{1}{n!} \cdot P_0} \qquad 0 \leq n \leq N_S$$

And that:

$$\boxed{P_{n+i} = \left(\frac{\lambda}{\mu n}\right)^{i} P_n} \qquad n \geq N_S$$

$n = N_S + i$

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^{n} \cdot \frac{1}{n!} P_0 & 0 \leq n \leq N_S \\ \left(\frac{\lambda}{\mu}\right)^{n-N_S?} \frac{1}{N_S!\, N_S^{\,n-N_S}} P_0 & n \geq N_S \end{cases}$$

$\hookrightarrow$ # times above N_S

By the NORMALIZATION CONDITION: $\boxed{\sum_{i=0}^{+\infty} p_i = 1}$

$$p_0 \sum_{n=0}^{N_S-1} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + p_0 \sum_{n=N_S}^{+\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{N_S!\, N_S^{\,n-N_S}} = 1$$

$$p_0 \cdot \left[\sum_{n=0}^{N_S-1} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \sum_{n=N_S}^{+\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{N_S! \cdot N_S^{\,n-N_S}}\right] = 1$$

$$\Rightarrow p_0 = \frac{1}{\sum_{n=0}^{N_S-1} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \sum_{n=N_S}^{+\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{N_S!\, N_S^{\,n-N_S}}}$$

$\frac{\lambda}{\mu} <$

$\Rightarrow \lambda < \mu$

CONDITIONS

Now consider this term for the ERGODICITY CONDITION!

$$\frac{1}{N_S!} \sum_{n=N_S}^{+\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{N_S^{\,n-N_S}} = \frac{1}{N_S!} \left(\frac{\lambda}{\mu}\right)^{N_S} \sum_{n=N_S}^{+\infty} \frac{\left(\frac{\lambda}{\mu}\right)^{n-N_S}}{N_S^{\,n-N_S}}$$

$$\Rightarrow \frac{1}{N_S!} \left(\frac{\lambda}{\mu}\right)^{N_S} \sum_{n=N_S}^{+\infty} \frac{\left(\frac{\lambda}{\mu}\right)^{n-N_S}}{N_S^{\,n-N_S}} = P_{N_S} \left(\frac{\lambda}{\mu}\right)^{n-N_S}$$

For $n - N_S = j$:

$$\sum_{j=0}^{+\infty} \left(\frac{\lambda}{N_S \cdot \mu}\right)^j = \frac{1}{1 - \frac{\lambda}{N_S \cdot \mu}} \qquad \sum_{i=0}^{+\infty} \alpha^i = \frac{1}{1-\alpha} \quad \text{if } |\alpha| < 1$$

ERGODICITY CONDITION:

$$\frac{\lambda}{\mu \cdot N_S} < 1 \Rightarrow \lambda < \mu \cdot N_S \Rightarrow \frac{\lambda}{\mu} < N_S \Rightarrow \boxed{A < N_S}$$

1) PROOF of $E\{T_W\}$ and $E\{T_W\}$ by PASTA PROPERTY in an M/M/Ns QUEUE

$$E\{T\} = E\{T_W\} + E\{T_S\} \quad E\{T_S\} = \frac{1}{\mu}$$

$$\underbrace{E\{T_W\}}_{\text{Waiting time}} = \sum_{k=N_S}^{+\infty} E\{T_W|k\}\, P_k^{(arr)}$$

Waiting time, where we have a queue formed (i.e. more customers than servers)

For 2 servers $(N_S = 2)$ [EXP. DISTRIBUTION & INDEPENDENT]

$$P\{T > t\} = P\{\min(T_1, T_2) > t\}$$
$$= P\{T_1 > t, T_2 > t\} = e^{-\mu_1 t} \cdot e^{-\mu_2 t} = e^{-(\mu_1+\mu_2)t}$$

$\Rightarrow$ We now want to find $\boxed{E\{T_W|k\}}$

$$\boxed{E\{T_W|N_S\} = \frac{1}{N_S\mu}}$$

(N_S customers already in the queue when arriving to it) $\Rightarrow$ Need to wait for them to be served

$$E\{T_W|N_S+1\} = \frac{2}{N_S\mu} = \frac{1}{N_S\mu} + \frac{1}{N_S\mu}$$

(N_S+1 customers already in the queue when arriving to it) $\Rightarrow$ Need to wait for them to be served

$$\boxed{E\{T_W|\underbrace{N_S+n}\} = \sum_{i=0}^{n} \frac{1}{N_S\mu} = \frac{(n+1)}{N_S\mu}}$$

$\Rightarrow N_S + n = k \Rightarrow n = k - N_S$

$$E\{T_W|k\} = \frac{k - N_S + 1}{N_S\mu} \qquad k \geq N_S$$

$$\boxed{E\{T_w \mid k\} = \frac{k - N_s + 1}{N_s \cdot \mu}} \qquad k \geq N_s$$

$$E\{T_w\} = \sum_{k=N_s}^{+\infty} E\{T_w \mid k\} \cdot P_k^{(arr)}$$

$$= \sum_{k=N_s}^{+\infty} \frac{k - N_s + 1}{N_s \mu} P_k^{(arr)} \qquad k \geq N_s$$

(evaluated before)

PASTA PROPERTY

$$= \sum_{k=N_s}^{+\infty} \frac{k - N_s + 1}{N_s \mu} P_{N_s} \cdot \left(\frac{\lambda}{\mu \cdot N_s}\right)^{k - N_s}$$

$$= \frac{P_{N_s}}{N_s \cdot \mu} \sum_{k=N_s}^{+\infty} (k - N_s + 1) \cdot \left(\frac{\lambda}{\mu \cdot N_s}\right)^{k - N_s}$$

Now set $\boxed{i = k - N_s + 1}$

$\boxed{i - 1 = k - N_s}$

$$= \frac{P_{N_s}}{N_s \mu} \sum_{i=1}^{+\infty} i \cdot \left(\frac{\lambda}{\mu N_s}\right)^{i-1}$$

$$= \frac{P_{N_s}}{N_s \mu \left(\frac{\lambda}{N_s \mu}\right)} \cdot \sum_{i=1}^{+\infty} i \cdot \left(\frac{\lambda}{\mu N_s}\right)^{i}$$

$$= \frac{P_{N_s}}{N_s \mu \cancel{\left(\frac{\lambda}{N_s \mu}\right)}} \cdot \frac{\cancel{\left(\frac{\lambda}{N_s \mu}\right)}}{\left(1 - \frac{\lambda}{N_s \mu}\right)^2}$$

$$= \frac{P_{N_s}}{N_s \mu \cdot \left(1 - \frac{\lambda}{N_s \mu}\right)^2}$$

$$= \frac{PNS}{NS\mu \cdot \left(\frac{NS\mu - \lambda}{NS\mu}\right)^2} = \frac{PNS}{\cancel{NS\mu} \cdot \frac{(NS\mu - \lambda)^2}{\cancel{(NS\mu)^2}}} \cdot \frac{NS\mu}{NS\mu}$$

$$= \frac{PNS \cdot NS\mu}{(NS\mu - \lambda)^2} = \frac{PNS \cdot NS \cdot \mu}{(NS\mu - \lambda)^2}$$

$$\Rightarrow E\{T_W\} = \frac{PNS \cdot NS \cdot \mu}{(NS\mu - \lambda)^2}$$

$$\Rightarrow E\{T\} = E\{T_W\} + E\{T_S\}$$

service time of one server

$$= \frac{PNS \cdot NS \cdot \mu}{(NS \cdot \mu - \lambda)^2} + \frac{1}{\mu}$$

NB:
N-free
ONLY
APPROACH.

No need to multiply by

$$\left(\frac{1}{NSM}\right)^{NS}$$

just:

$$\sum_{n=NS}^{\infty} \alpha^n = \frac{(\alpha)^{NS}}{1-\alpha}$$

48

ERLANG - C FORMULA (for $\boxed{M/M/N_s}$ QUEUES)

Used to estimate the PROBABILITY OF EXPERIENCING DELAY $\boxed{P_D}$

Easier to evaluate in EXOGENEOUS SYSTEMS, where:

$$\boxed{P_D = P_{BS}}$$

$$\boxed{P_K = \left(\frac{\lambda}{\mu}\right)^K \frac{1}{N_S!} \frac{1}{N_S^{K-N_S}} P_0} \quad \boxed{K \geq N_S}$$

$$P_D = P_{BS} = \sum_{K=N_S}^{+\infty} P_K = \sum_{K=N_S}^{+\infty} P_0 \left(\frac{\lambda}{\mu}\right)^K \frac{1}{N_S!} \frac{1}{N_S^{K-N_S}}$$

$$= \sum_{K=N_S}^{+\infty} P_0 \left(\frac{\lambda}{\mu}\right)^K \frac{1}{N_S!} \frac{N_S^{N_S}}{N_S^K} = \sum_{h=0}^{+\infty} \left(\frac{\lambda}{\mu N_S}\right)^h \quad h = K - N_S$$

$$= \frac{N_S^{N_S}}{N_S!} \left(\frac{\lambda}{N_S \mu}\right)^{N_S} P_0 \sum_{K=N_S} \left(\frac{\lambda}{\mu N_S}\right)^{K-N_S}$$

CONVERGES for $\boxed{\lambda < N_S \mu}$

$$= \frac{\cancel{N_S^{N_S}}}{N_S!} P_0 \frac{\left(\frac{\lambda}{\cancel{N_S}\mu}\right)^{N_S}}{1 - \frac{\lambda}{N_S \mu}} \qquad \frac{\lambda}{N_S\mu} = \rho$$

AVERAGE AMOUNT of WORK accepted / work server

$$= \frac{P_0}{N_S!} \frac{\left(\frac{\lambda}{\mu}\right)^{N_S}}{1 - \rho}$$

Where $\rho = \frac{A}{N_S} = \frac{E[n_s]}{N_S}$

[BECAUSE NO LOSS] $\frac{A_0}{N_S} = \frac{\lambda}{\mu N_S}$

$A_0 = \frac{\lambda}{\mu} \Rightarrow \rho = \frac{A}{N_S}$

So far, we have found that

$$P_D = P_{BS} = \frac{P_0}{N_S!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{N_S}}{(1-\rho)}$$

$$P_k = \begin{cases} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} P_0 & k \le N_S \\ \left(\frac{\lambda}{\mu}\right)^{N_S} \frac{1}{N_S! \cdot N_S^{k-N_S}} \end{cases}$$

$$\Rightarrow P_D = \frac{\boxed{\frac{1}{N_S!} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^{N_S}}{(1-\rho)}}}{\sum_{k=0}^{N_S-1} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} + \boxed{\frac{1}{N_S!} \left(\frac{\lambda}{\mu}\right)^{N_S} \frac{1}{1-\rho}}}$$

$P_0 =$

Multiply by the inverse of the term!

$\rho = E[N_S]$

$\Rightarrow$ ERLANG - C FORMULA | 2nd TYPE ERLANG FORMULA

$$\boxed{A_0 = \frac{\lambda}{\mu}} \qquad \boxed{\rho = \frac{\lambda}{N_S \mu} = \frac{A_0}{N_S}} \qquad A = \text{TRAFFIC INTENSITY}$$

$$\boxed{P_D = \frac{1}{1 + \frac{N_S!}{A_0^{N_S}} \left(1 - \frac{A_0}{N_S}\right) \sum_{k=0}^{N_S-1} A_0^k \frac{1}{k!}}}$$

RECURSIVE ERLANG - C FORMULA

$$C_{N_S}(A_0) = \begin{cases} \dfrac{1 - \frac{A_0}{N_S-1}}{\dfrac{\frac{N_S}{A_0} - 1}{C_{N_S-1}(A_0)} - \dfrac{1}{N_S-1}} & \text{if } N_S > 1 \\ A_0 & N_S = 1 \end{cases}$$

is

For $\boxed{P_D}$ in M|M|N_S QUEUES

6/8

FULL STEPS FOR ERLANG-C FORMULA'S DEDUCTION:

$$P_D = P_{BS} = \frac{P_0}{N_S!} \frac{\left(\frac{\lambda}{\mu}\right)^{N_S}}{1-\rho}$$

We know that P_0 is:

$$P_0 = \frac{1}{\sum_{i=0}^{N_S-1} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} + \sum_{i=N_S}^{\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{N_S!} \frac{1}{N_S^{\,i-N_S}}}$$

$\Rightarrow$ PD is, substituting P_0 in it:

$$P_D = \frac{\dfrac{\left(\frac{\lambda}{\mu}\right)^{N_S}}{N_S!\,(1-\rho)}}{\sum_{i=0}^{N_S-1} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} + \sum_{i=N_S}^{\infty} \frac{\left(\frac{\lambda}{\mu}\right)^i \cdot N_S^{N_S}}{N_S! \cdot N_S^{\,i}}}$$

$$= \frac{\dfrac{\left(\frac{\lambda}{\mu}\right)^{N_S}}{N_S!\,(1-\rho)}}{\sum_{i=0}^{N_S-1} \left(\frac{\lambda}{\mu}\right)^i \cdot \frac{1}{i!} + \frac{N_S^{N_S}}{N_S!} \cdot \sum_{i=N_S}^{\infty} \left(\frac{\lambda}{N_S\mu}\right)^i}$$

$$\frac{N_S^{N_S}}{N_S!} \sum_{i=N_S}^{+\infty} \left(\frac{\lambda}{\mu \cdot N_S}\right)^i$$

$$= \frac{N_S^{N_S}}{N_S!} \frac{\left(\frac{\lambda}{\mu \cdot N_S}\right)^{N_S}}{\left(1 - \frac{\lambda}{\mu \cdot N_S}\right)} = \frac{\cancel{N_S^{N_S}} \left(\frac{\lambda}{\mu}\right)^{N_S} \cdot \frac{1}{\cancel{N_S^{N_S}}}}{N_S! \left(1 - \frac{\lambda}{\mu \cdot N_S}\right)}$$

For $\boxed{\frac{\lambda}{\mu N_S} = \rho}$; $\boxed{\frac{\lambda}{\mu} = A_0}$

$$= \frac{(A_0)^{N_S}}{N_S! \, (1-\rho)}$$

$$\Rightarrow P_D = \frac{\cancel{\frac{(A_0)^{N_S}}{N_S!\,(1-\rho)}}}{\sum_{k=0}^{N_S-1} (A_0)^k \frac{1}{k!} + \cancel{\frac{1}{N_S!} \cdot \frac{(A_0)^{N_S}}{(1-\rho)}}} \cdot \frac{\cancel{\frac{N_S!\,(1-\rho)}{(A_0)^{N_S}}}}{\frac{N_S!\,(1-\rho)}{(A_0)^{N_S}}}$$

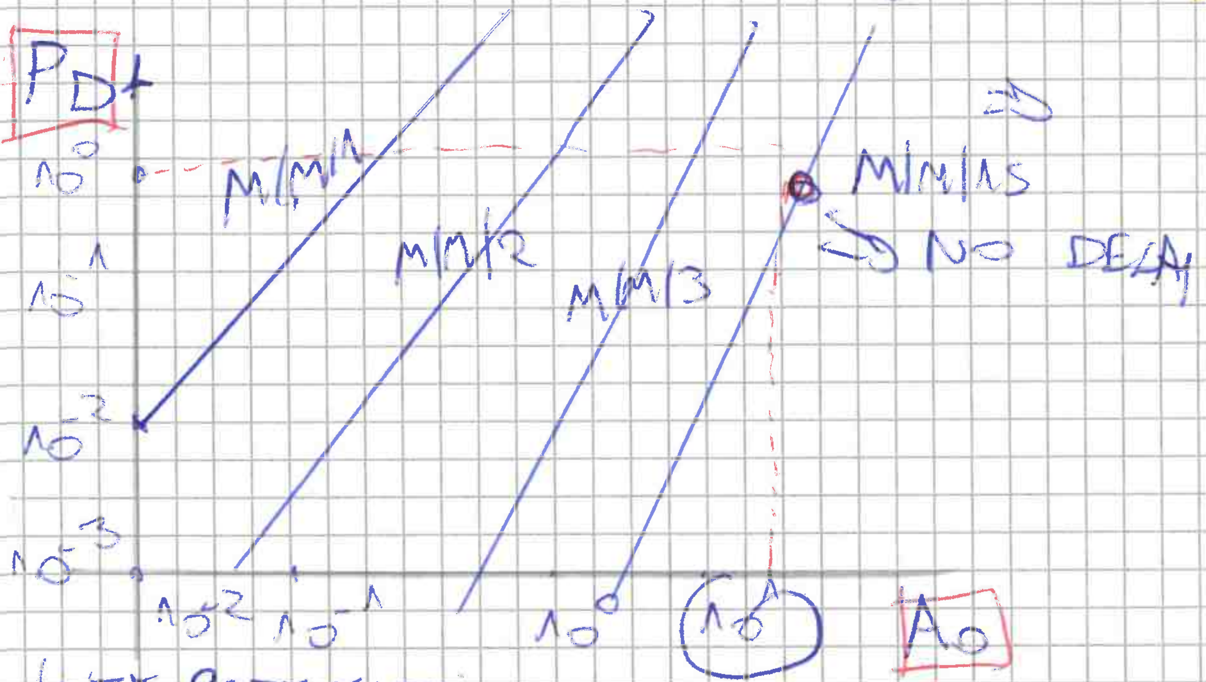
$$\Rightarrow P_D = \frac{1}{\frac{N_S!\,(1-\rho)}{(A_0)^{N_S}} \sum_{k=0}^{N_S-1} \frac{(A_0)^k}{k!} + 1}$$

ERLANG-C FORMULA

$\cdot\ \rho = \frac{A_0}{N_S}$

$$\boxed{P_D = \frac{1}{\frac{N_S!}{(A_0)^{N_S}} \left(1 - \frac{A_0}{N_S}\right) \sum_{k=0}^{N_S-1} (A_0)^k \frac{1}{k!}}}$$

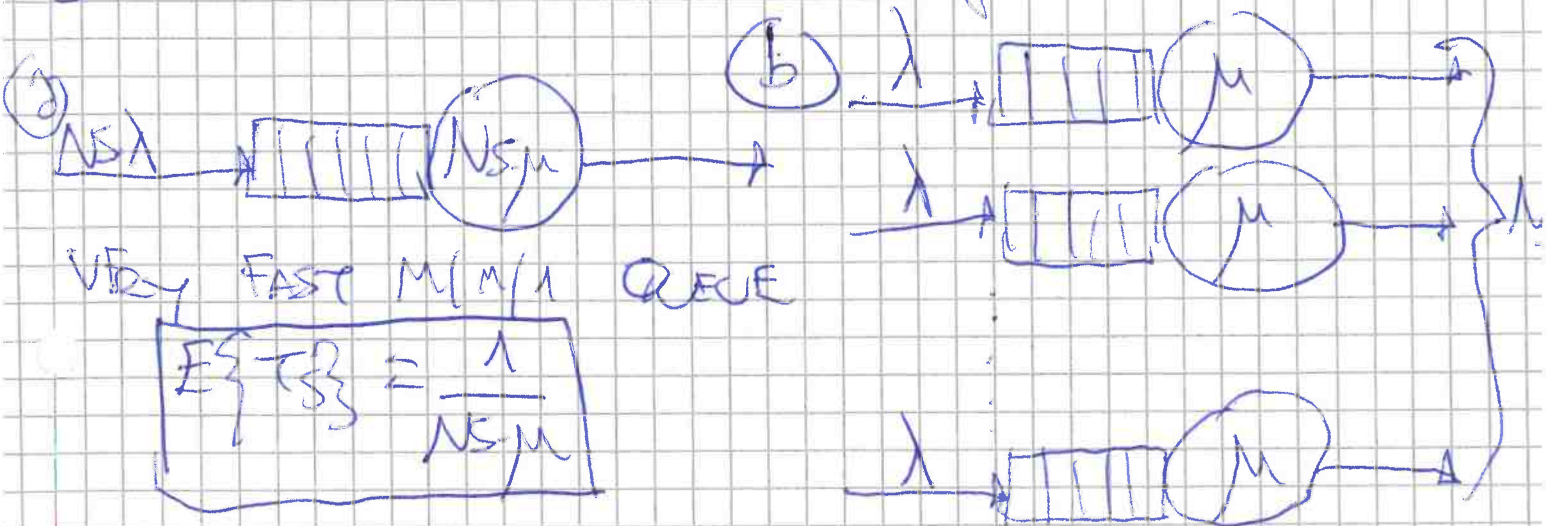
⇒ The Erlang-C formula is used to links together the traffic intensity A_0 with the PROBABILITY of DELAY P_D



INTERPRETATION:
"Given the offered load, what is the # SERVERS I need to use to obtain the desired P_D?
↳ 10 ERLANGS
↳ $A_0 = 10$

(4.9) PERFORMANCE COMPARISON BETWEEN:

a) M/M/1, One W.L. for ~~each~~ the whole one queue.
b) N_S - many M/M/1 QUEUES, one W.L. per queue.
c) M/M/N_S QUEUE, One W.L. for all N_S SERVERS



VERY FAST M/M/1 QUEUE

$$\boxed{E\{T_S\} = \frac{1}{N_S\mu}}$$

b) Ns-many M/M/1 QUEUES.

$$\boxed{E\{T_S\} = \frac{1}{\mu}}$$

Ns = #SERVERS

c) M/M/Ns QUEUE:

One waiting line for all Ns-many servers.

$$\boxed{E\{T_S\} = \frac{1}{\mu}}$$

PERFORMANCE COMPARISON:

$$\boxed{E\{T\} = \frac{\frac{1}{\mu}}{1-\rho}} \quad \boxed{M/M/1} \quad (A)(b)$$

$$\boxed{E\{T\} = \underbrace{\frac{1}{\mu}}_{E\{T_S\}} + \underbrace{\frac{Ns\mu \cdot P_{Ns}}{(Ns\mu - \lambda)^2}}_{E\{T_W\}}} \quad \boxed{M/M/Ns} \quad (C)$$

$$\boxed{E\{T^{(a)}\}} = M/M/1 \text{ with } \frac{\lambda}{Ns\mu} \text{ RATE}$$

$$E\{T^{(a)}\} = \frac{1}{Ns\mu}\,\frac{1}{1-\rho} = \frac{1}{Ns\mu}\,\frac{1}{1-\frac{\cancel{Ns}\lambda}{\cancel{Ns}\mu}}$$

$$\Rightarrow E\{T^{(a)}\} = \frac{1}{\cancel{\mu}Ns\cdot\frac{(\mu-\lambda)}{\cancel{\mu}}} = \frac{1}{Ns\cdot(\mu-\lambda)}$$

A: $\rho < 1$

$Ns\,\lambda < Ns\,\mu$

$$\Rightarrow \boxed{E\{T^{(a)}\} = \frac{1}{Ns\,(\mu-\lambda)}}$$

$E\{T^{(b)}\}$ = For each M|M|1 QUEUE, we have:

$$E\{T^{(b)}\} = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu(1-\frac{\lambda}{\mu})} = \frac{1}{\frac{\mu(\mu-\lambda)}{\mu}} = \frac{1}{\mu-\lambda}$$

$$\Rightarrow \boxed{E\{T^{(b)}\} = \frac{1}{\mu-\lambda}}$$

ERGODICITY CONDITION: $\lambda < \mu$

$E\{T^{(c)}\}$ = M|M|NS QUEUE

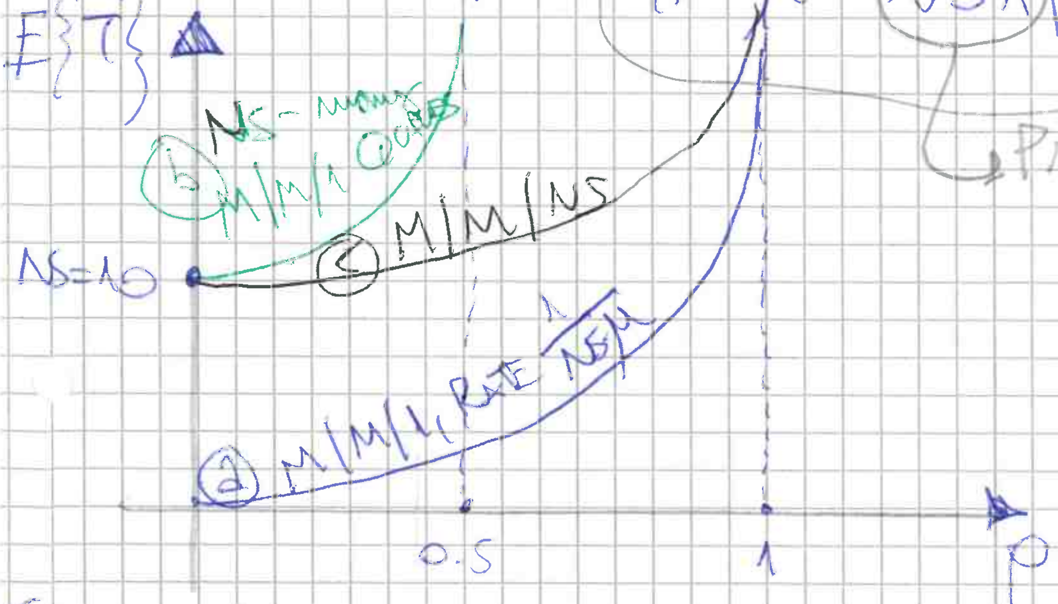
$$E\{T^{(c)}\} = E\{T_S\} + E\{T_W\}$$

$$= \frac{1}{\mu} + \frac{N_S \cdot \mu \cdot P_{NS}}{(N_S\mu - N_S\lambda)^2}$$

Found by the PASTA PROPERTY in 47 (d)

ERGODICITY CONDITION: $N_S\lambda < N_S\mu$

Previously, we had λ (now arrival rate is different)



(a), "SPEEDY GONZALEZ" M|M|1 QUEUE is NUMBER ONE (LIKE CHINA) (a)

(b) NS-many M|M|1 QUEUES is NS-times SPEEDY GONZALEZ slower than M|M|1 QUEUE with a rate NS

UNLOADED SITUATION ($\rho \to 0$)

$$E\{T^{(b)}\} \approx E\{T^{(c)}\} \approx \frac{1}{\mu}$$

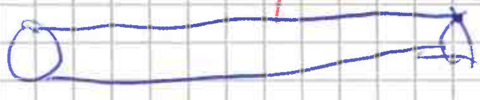
INCREASED-LOAD SITUATION:

N_S - many M/M/1 QUEUES (b) $\ll$ (SLOWER THAN) M/M/N_S (c)

⇒ Some queues (& waiting lines) from (b) may be IDLE ⇒ wasted IDLE SERVERS.

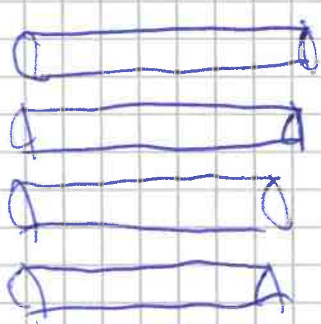
⇒ (c) is better!

IN CONCLUSION, we prefer ONE BIG FAT LINK ⇐ ECONOMY of SCALE!



2Mb/s

rather than 32 links at 64 Kb/s, each.



64 Kb/s

64 Kb/s

64 Kb/s

Also, we'd prefer to have one Co for many customers rather than one small Co for a few customers.

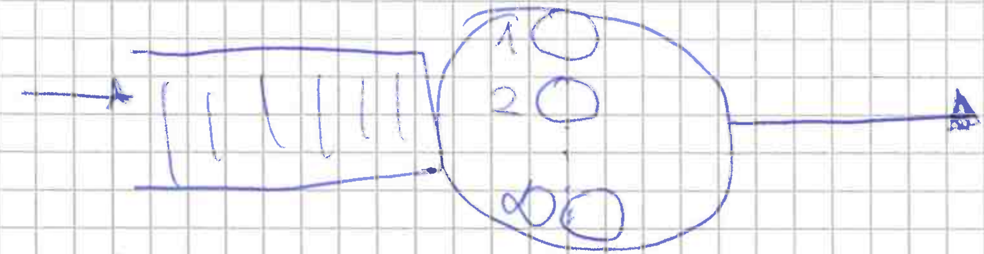
So M/M/∞ QUEUE, state probability P_K

$N_S = \infty$ [Infinite # SERVERS]

⇓

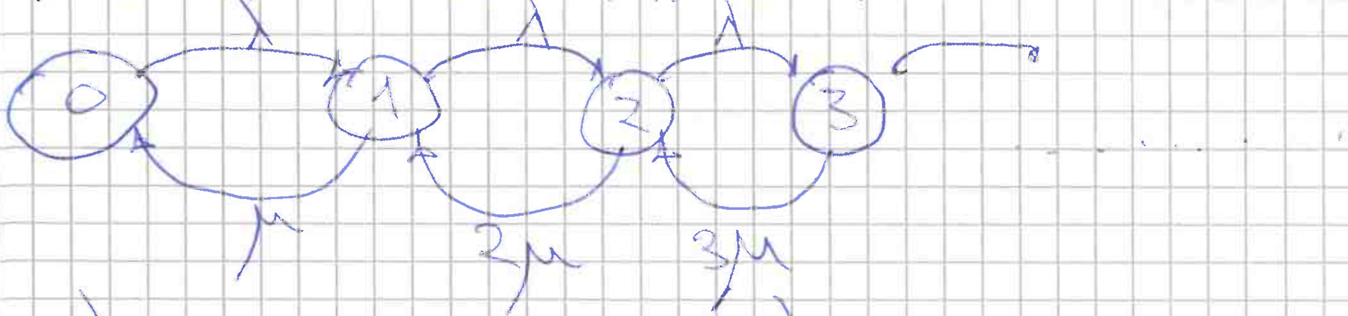
Customers always find one AVAILABLE SERVER.

⇒ No waiting! $E\{T_W\} = 0$



M|M|∞ QUEUE

RATE TRANSITION DIAGRAM:



$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0$

$\lambda p_1 = 2\mu p_2 \Rightarrow p_2 = \frac{\lambda}{2\mu} p_1 = \left(\frac{\lambda}{\mu}\right)^2 \cdot \frac{1}{2} p_0$

$\lambda p_2 = 3\mu p_3 \Rightarrow p_3 = \frac{\lambda}{3\mu} p_2 = \left(\frac{\lambda}{\mu}\right)^3 \cdot \frac{1}{6} p_0$

$$\Rightarrow \boxed{p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} p_0} \qquad n \geq 0$$

By the NORMALIZATION CONDITION: $\sum_{i=0}^{+\infty} p_i = 1$

$$p_0 \sum_{i=0}^{+\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!} = 1 \Rightarrow p_0 = \frac{1}{\underbrace{\sum_{i=0}^{+\infty} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}}_{= e^{\frac{\lambda}{\mu}}}}$$

$$\Rightarrow \boxed{p_0 = e^{-\frac{\lambda}{\mu}}}$$

$$\boxed{\phi = \frac{\lambda}{\mu}}$$

$$\Rightarrow \boxed{p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} \cdot e^{-\frac{\lambda}{\mu}}} = (\phi)^n \cdot \frac{1}{n!} \cdot e^{-\phi}$$

$$\Rightarrow p_n = \frac{(\phi)^n \, e^{-\phi}}{k!}$$ POISSON DISTRIBUTION FOR A DISCRETE RV.

$E\{T_w\} = 0$

$E\{T\} = \frac{1}{\mu} = E\{T_s\}$

$\Rightarrow E\{n_w\} = 0 \quad \Rightarrow E\{n\} = E\{n_s\} = A = \lambda \cdot E\{T_s\}$

$= \frac{\lambda}{\mu}$

$$\boxed{\rho = \frac{E\{n_s\}}{N_s \to \infty} = 0}$$

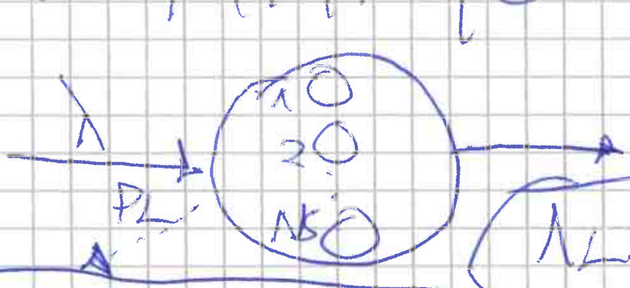
EXTREMELY LOW UTILIZATION FACTOR

(1) PROOF OF ERLANG-B FORMULA, TO FIND P_L IN AN $M|M|N_S|0$ QUEUE

Consider an $M|M|N_S|0$ QUEUE

$N_S = \#$ SERVERS

$N_W = 0$

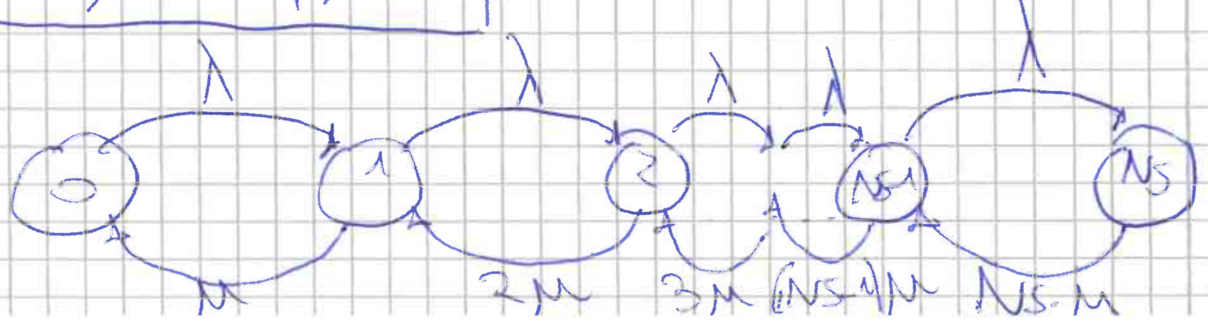


$\lambda_L = \lambda_0 \cdot P_L$

$$\boxed{P_k = \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} p_0}$$

$0 \le k \le N_S$

RGF



$$\boxed{P_0 = \frac{1}{\sum_{i=0}^{N_S} \left(\frac{\lambda}{\mu}\right)^i \cdot \frac{1}{i!}}}$$

By the NORMALIZATION CONDITION.

(After PK) was obtained for M|M|1|NS QUEUE

$$\Rightarrow P_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} \cdot P_0 \qquad 0 \le n \le N_S$$

$$\frac{1}{P_0} = \sum_{n=0}^{N_S} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!}$$

ERLANG FORMULA of LOSS - TYPE (B FORMULA)

$$\boxed{\Rightarrow P_L = P_B = P_{N_S} = \frac{\left(\frac{\lambda}{\mu}\right)^{N_S} \frac{1}{N_S!}}{\sum_{n=0}^{N_S} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}}}$$

P_N

$N = N_S + N_Q$, $N_Q = 0$

Such formula is used to model CIRCUIT - SWITCHED NETWORK (i.e. the telephone).

$$\Lambda = \Lambda_0 (1 - P_L) = \lambda (1 - P_{N_S}) = \Gamma$$

$$A = A_0 (1 - P_L) = \frac{\lambda}{\mu} (1 - P_{N_S}) = E\{n_S\} = E\{n\}$$

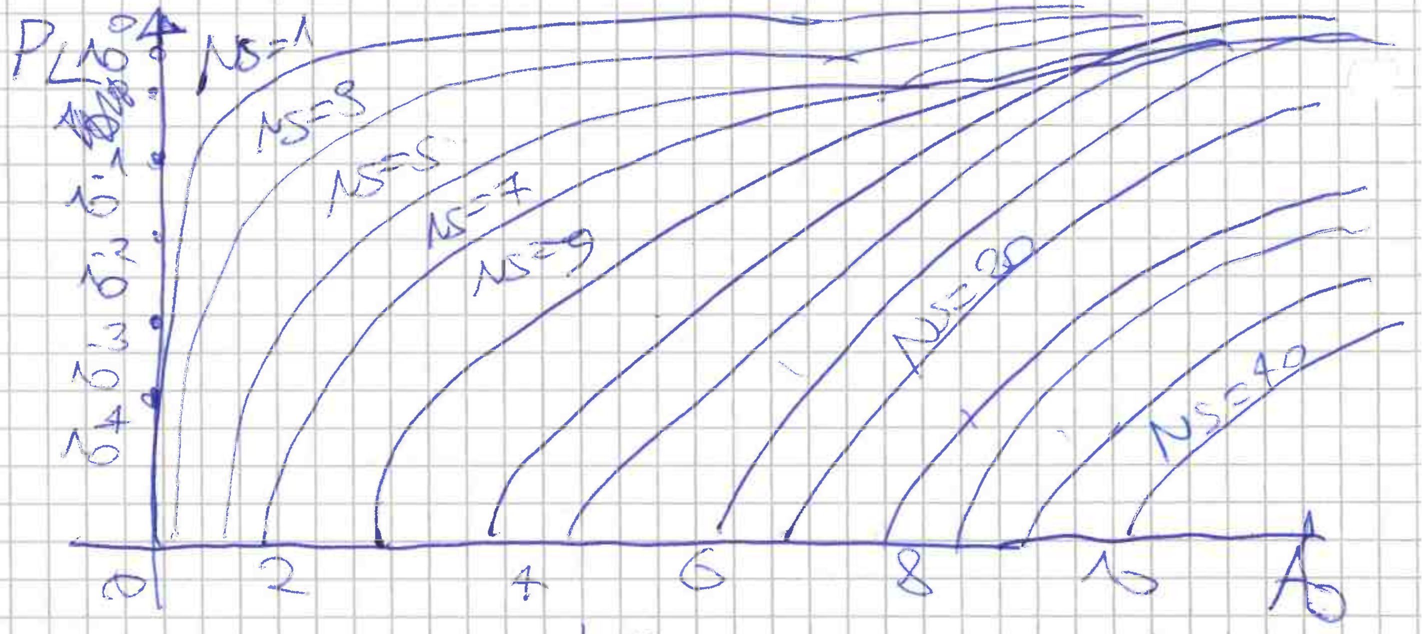
A_0: OFFERED TRAFFIC INTENSITY, ERLANGS.

$$\rho = \frac{E\{n_S\}}{N_S} = \frac{A}{N_S} = \frac{\lambda (1 - P_{N_S})}{\mu \cdot N_S}$$

RECURSIVE ERLANG-B FORMULA

$$\boxed{B_{N_S}(A_0) = \begin{cases} \dfrac{A_0}{A_0 + \dfrac{N_S}{B_{N_S-1}(A_0)}} & \text{if } N_S \ge 1 \\ 1 & \text{if } N_S = 0 \end{cases}}$$

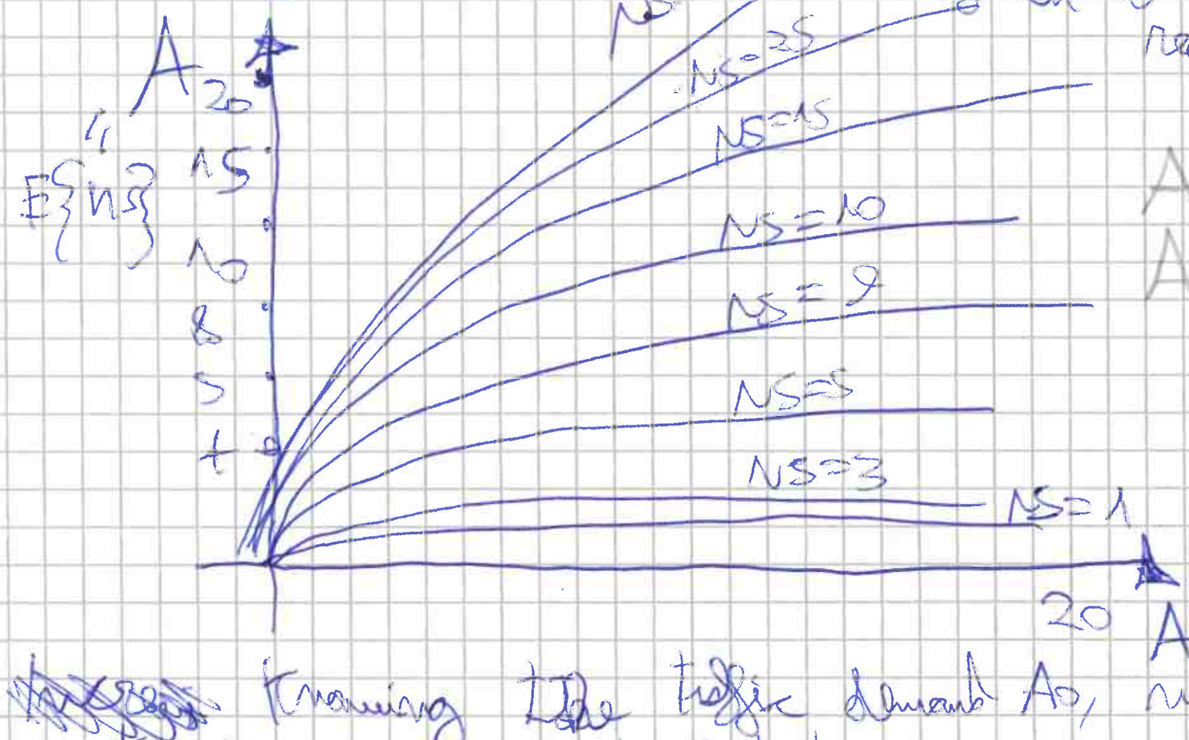
3 PLOTS :



Increasing the offered traffic intensity A_0 (ERLANG), we need to increase the #SERVERS NS to "contain the P_L". The P_L increases by increasing A_0. ⇒ Can determine #SERVERS (NS) given A_0 & traffic requirements & P_L.

$A = E\{ns\}$

$A_0 = \frac{\lambda}{\mu}$

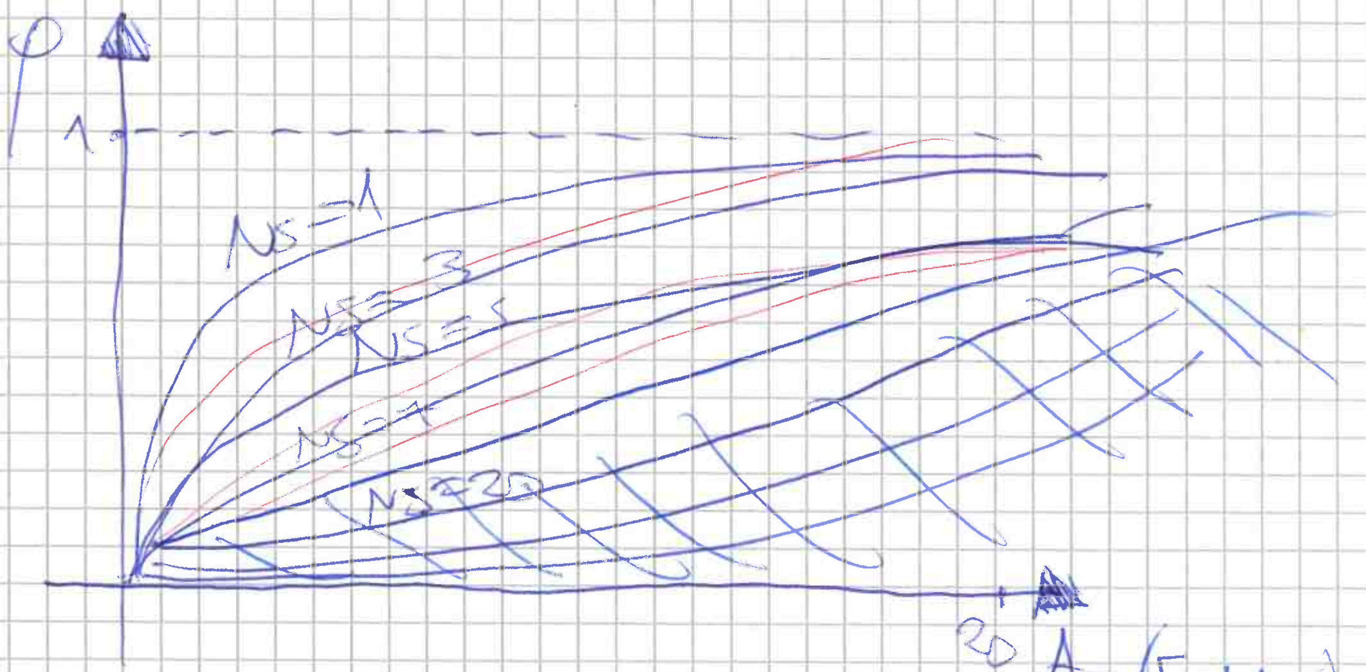


~~[illegible]~~ Knowing the traffic demand A_0, we can determine the #SERVERS NS to find the ACCEPTED TRAFFIC A.

UTILIZATION COEFFICENT ρ

VS

OFFERED TRAFFIC in M|M|Ns|0 QUEUE.



Increasing the # SERVERS & keeping the same A0, the UTILIZATION FACTOR DECREASES.

~~(B) RECURSIVE FORM of ERLANG.~~

(C) PROPERTY of INSENSIBILITY of ERLANG-B FORMULA

The formula (Erlang-B) is insensible to the PDF of the $\boxed{\text{SERVICE TIME}}$, as long as certain hypotheses are met.

3 $\boxed{\text{REQUIRED HYPOTHESES.}}$

(1) POISSON ARRIVALS

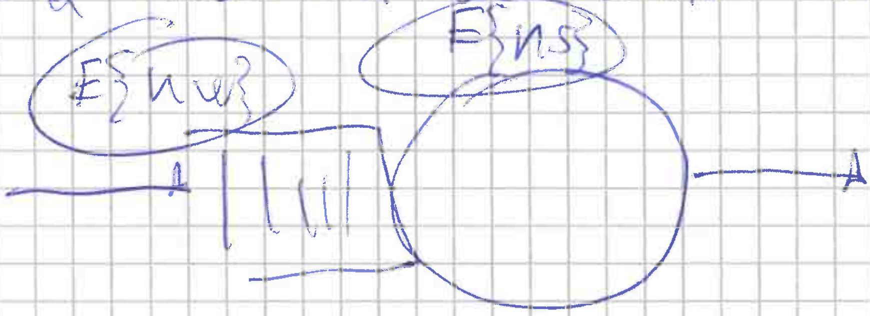
(2) FINITE MOMENTUM of SERVICE-TIME DISTRIBUTION.

$\hookrightarrow$ heavy-tail distribution $\Rightarrow$ $\boxed{VAR\{x\} = \infty}$

(3) EXPONENTIAL INTERARRIVAL TIMES (& INDEPENDENT)

(52) LITTLE'S FORMULA PROOF

We want to show that, at STEADY-STATE, for a CONSERVATIVE SYSTEM we have:



$$E\{n\} = \Lambda \cdot E\{T\}$$

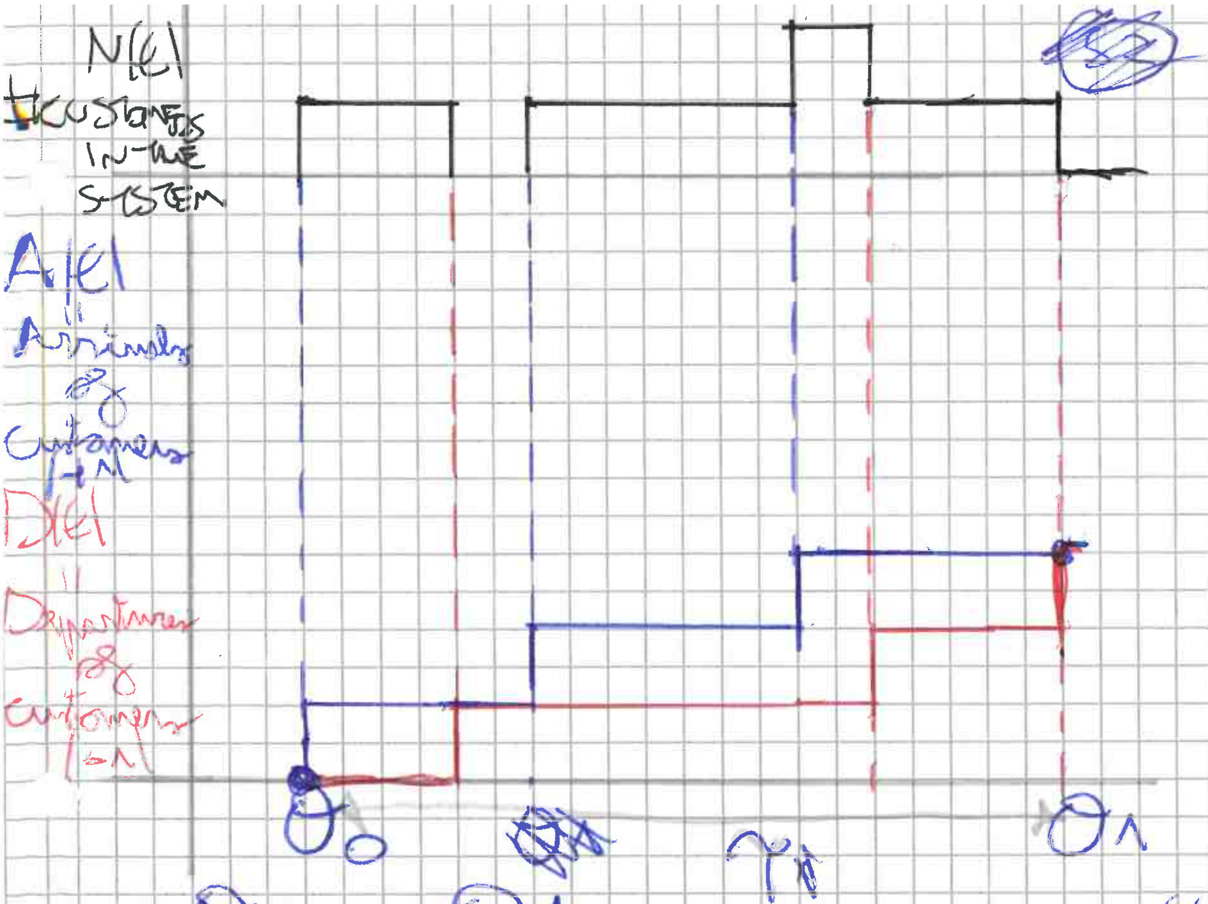
$$E\{n_w\} = \Lambda \cdot E\{T_w\}$$

$$E\{n_s\} = \Lambda \cdot E\{T_s\}$$

ACCEPTED TRAFFIC INTENSITY

Where $E\{n\} = E\{n_w\} + E\{n_s\}$

PROOF:



$\theta_0 - \theta_1 =$ REGENERATIVE POINTS = Points (in time) with NO CUSTOMERS in the SYSTEM.

$\boxed{A(t)}$ = # ARRIVED CUSTOMERS in (θ_0, θ_0+t)

$\boxed{D(t)}$ = # DEPARTED CUSTOMERS in (θ_0, θ_0+t)

$\boxed{A(\theta_0) = D(\theta_0)}$. $\boxed{A(\theta_1) = D(\theta_1)}$ EMPTY SYSTEM

N(t) = # CUSTOMERS in the queue at time t.

$$\boxed{N(t)} = A(t) - D(t)$$

REGENERATIVE POINTS.

$N(\theta_0) = 0, \quad N(\theta_1) = 0$

INITIALLY, we have an empty SYSTEM, and we have

$$\boxed{\tau = \theta_1 - \theta_0}$$

RECALL what LITTLE'S FORMULA relates:

(1) # CUSTOMERS in the system $E\{n\}$

(2) AVERAGE RATE of ARRIVALS Λ

(3) AVERAGE time spent in the system $E\{T\}$

$E\{n\} = \Lambda E\{T\}$

(1) $E\{n\} \to \bar{n} = \frac{1}{\tau} \int_{\theta_0}^{\theta_1} N(t)\, dt$

(3) $E\{T\} \to \boxed{\bar{w} = \frac{1}{A(\tau)} \sum_{j=1}^{A(\tau)} w_j}$ → AVERAGE time spent in the queue by the j-th customer

ARRIVED CUSTOMERS

ARRIVALS in τ

(2) $\boxed{\Lambda} \to \boxed{\bar{\lambda}} = \frac{A(\tau)}{\tau} \Rightarrow \boxed{A(\tau) = \bar{\lambda} \cdot \tau}$

= # ARRIVALS over interval

τ = interval duration τ

Now putting these concepts together (i.e. $A(\tau)$ into $\bar{w}$)

(3'): $\boxed{\bar{w}' = \frac{1}{\bar{\lambda} \cdot \tau} \sum_{j=1}^{A(\tau)} w_j}$

$A(\tau) = \bar{\lambda} \cdot \tau$

Now consider the CHARACTERISTIC FUNCTION

Has value (1) only if there is a customer INSIDE the queue.

CHARACTERISTIC FUNCTION $I_j(t)$

$$I_j(t) = \begin{cases} 1 & a_j \le t \le d_j \\ 0 & \text{elsewhere} \end{cases}$$

Sum of the char function of all customers

$$N(t) = \sum_{j=1}^{A(t)} I_j(t) = \sum_{j=1}^{A(\tau)} I_j(t)$$

$\Rightarrow$ WHOLE PERIOD

{ Overall # customers in the system }

$I_j(t) = 0$ when no customers are present

$$\boxed{w_j = \int_{\theta_0}^{\theta_1} I_j(t)\, dt}$$

(1) $\bar{n} = \frac{1}{\tau} \int_{\theta_0}^{\theta_1} \sum_{j=1}^{A(\tau)} I_j(t)\, dt$

(3) $\bar{w}^* = \frac{1}{\bar{\lambda} t} \sum_{j=1}^{A(\tau)} w_j$

$$\bar{n} = \bar{\lambda} \cdot \bar{w}^* = \frac{1}{\cancel{\bar{\lambda}} \tau} \sum_{j=1}^{A(\tau)} \int_{\theta_0}^{\theta_1} I_j(t)\, dt = \cancel{\bar{\lambda}}$$

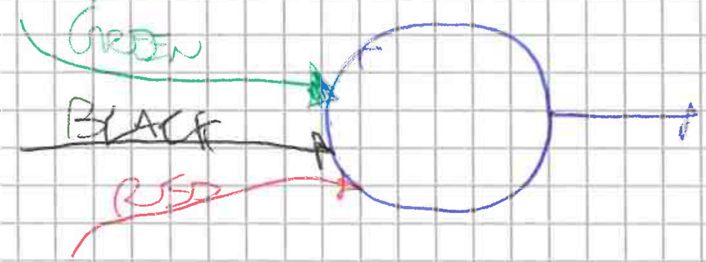
$\Rightarrow$ We can hence conclude that

$$\bar{\lambda} \cdot \bar{w} = \bar{n} \quad \text{(LITTLE'S FORMULA)}$$

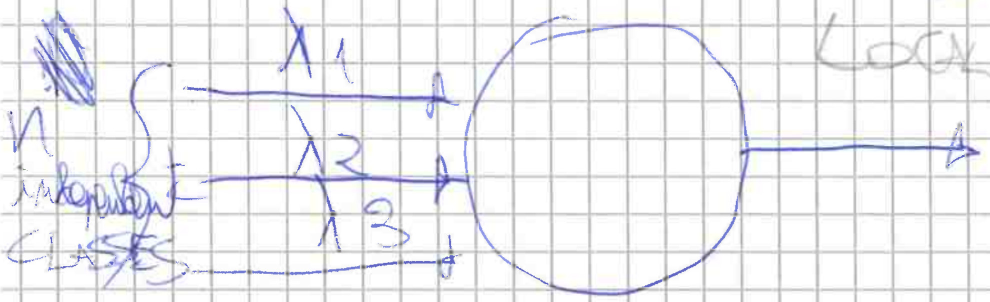
$$\Rightarrow \boxed{E\{n\} = \Lambda \cdot E\{\tau\}}$$

~~(53) DEFINITION of EMBEDDED (HIDDEN) MARKOFF CHAIN.~~

(55) DEFINITION of LOCAL & LOCAL $E\{T_{w,i}\}$, $E\{n_i\}$, $E\{n_{s,i}\}$, $E\{n_{w,i}\}$, NO PRIORITY BUT MULTIPLE USER CLASSES



3 different user classes, yet all of them have the same PRIORITY.



LOCAL ⇒ for one CLASS

GLOBAL ⇒ for all classes

$$\boxed{\lambda = \sum_i \lambda_i}$$

One class is independent from the other ones.

$$\boxed{f_{T_S}(t) = \frac{1}{\lambda} \cdot \sum_{i=1}^{n} \lambda_i \cdot f\{T_{S,i}\}}$$

GLOBAL PDF of the SERVICE PDF.

$$\boxed{E\{T_S\} = \frac{1}{\lambda} \cdot \sum_{i=1}^{n} \lambda_i \cdot E\{T_{S,i}\}}$$

GLOBAL $E\{T_S\}$

$$\rho = \frac{E\{n_S\}}{1} = \lambda \cdot E\{T_S\} = \cancel{\lambda} \cdot \frac{1}{\cancel{\lambda}} \sum_{i=1}^{n} \lambda_i E\{T_{S,i}\}$$

$$= \sum_{i=1}^{n} \underbrace{\lambda_i \cdot E\{T_{S,i}\}}_{=\rho_i} = \sum_{i=1}^{n} \rho_i$$

$$\boxed{\rho = \sum_{i=1}^{n} \rho_i} \Rightarrow \rho < 1 \quad N_S$$

GLOBAL UTILIZATION FACTOR & ERGODICITY CONDITION.

$$E\{T_W\} = \frac{\lambda \cdot E\{T_S^2\}}{2(1-\rho)} = \frac{\lambda \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} E\{T_{S_i}^2\}}{2(1-\sum_{i=1}^{n} \rho_i)}$$

By the P-K FORMULA

$$= \frac{\sum_{i=1}^{n} \lambda_i E\{T_{S_i}^2\}}{2(1-\sum_{i=1}^{n} \rho_i)}$$

Need to consider all the other classes for each class $E\{T_{W_i}\}$

$$\Rightarrow \boxed{E\{T_{W_i}\} = E\{T_W\}}$$

GLOBAL $E\{T_W\}$ = LOCAL $E\{T_{W_i}\}$

$$\forall \quad \boxed{E\{T_i\} = E\{T_W\} + E\{T_{S_i}\}}$$

LOCAL QUEUEING TIME

$$\boxed{E\{T\} = E\{T_W\} + E\{T_S\}}$$

GLOBAL QUEUEING TIME

$$\boxed{E\{T\} = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i \cdot E\{T_i\}}$$

AVERAGE # CUSTOMERS ($E\{n_W\}$, $E\{n\}$, $E\{n_S\}$)

↳ Cumulative METRIC!

$$\boxed{E\{n_{W_i}\} = \lambda_i \cdot E\{T_{W_i}\}} = \lambda_i \cdot E\{T_W\}$$

$$\boxed{E\{n_W\} = \sum_{i=1}^{n} E\{n_{W_i}\}}$$

GLOBAL # CUSTOMERS IN WAITING LINE

$\cancel{E\{n_i\} = \lambda \cdot E\{T_i\} =}$

$$E\{n_w\} = \sum_{i=1}^{n} E\{n_{wi}\} = \lambda \cdot E\{T_w\} = \frac{\lambda^2 \cdot E\{T_S^2\}}{2(1-\rho)}$$

$$E\{n_i\} = \lambda_i \cdot E\{T_i\} = \underbrace{E\{n_{Si}\}}_{\rho_i} + E\{n_{wi}\}$$

$$\Rightarrow \boxed{E\{n_i\} = \rho_i + E\{n_{wi}\}}$$ PER-CLASS GLOBAL # CUSTOMERS

$$\boxed{E\{n\} = \sum_{i=1}^{n} E\{n_i\} = \rho + E\{n_w\}}$$ GLOBAL # CUSTOMERS in all QUEUES

$$\boxed{E\{n\} = \rho + \frac{\lambda^2 \cdot E\{T_S^2\}}{2(1-\rho)}}$$

(56) DEFINITION of VIRTUAL & RESIDUAL TIME with NO PRIORITY CLASSES (WITH USER CLASSES!)

$$\boxed{E\{T_w\} = E\{T_V\} + E\{T_R\}}$$

VIRTUAL TIME (Service time of all arrived customers before me in the waiting line)

RESIDUAL SERVICE TIME: Remaining service time of the customer being served as I arrive. (Non-Markovian nature of the queue)

$E\{T_v\} = \sum_{i=1}^{K} E\{T_{si}\} = E\{T_{si}(1) + T_{si}(2) + \dots + T_{si}(K)\}$

$$\boxed{E\{T_v\} = K \cdot E\{T_s\}}$$

customers in the WAITING LINE

$E\{E\{T_{si}^{(1)} + T_{si}^{(2)} + \dots + T_{si}(K) \mid N_{wi} = K_i\}\}$

$$\Rightarrow \boxed{E\{T_v\} = \sum_{i=1}^{n} \underbrace{E\{N_{wi}\}}_{\lambda_i E\{T_{wi}\}} \cdot E\{T_{si}\}}$$

$\cancel{= \sum_{i=1}^{n} E\{N_{wi}\} \cdot \lambda}$

$\cancel{= \sum_{i=1}^{n} \lambda_i E\{T_{wi}\} \quad \rho_i}$

$\Rightarrow E\{T_v\} = \sum_{i=1}^{n} \lambda_i \cdot E\{T_{wi}\} \, E\{T_{si}\} \quad = \rho_i$

$= \sum_{i=1}^{n} \rho_i \, E\{T_{wi}\}$

VIRTUAL TIME

$$\Rightarrow \boxed{E\{T_v\} = \rho \cdot E\{T_w\}} = \sum_{i=1}^{n} \rho_i \cdot E\{T_{wi}\}$$

By the P-K FORMULA:

$$\boxed{E\{T_v\} = \frac{\lambda E\{T_s^2\}}{2(1-\rho)} \cdot \rho}$$

$\Rightarrow$ Now we can find $E\{T_R\} = E\{T_w\} - E\{T$

$$E\{T_R\} = E\{T_W\} - E\{T_V\}$$
$$= E\{T_W\} - \rho \cdot E\{T_W\}$$

$$\Rightarrow E\{T_R\} = E\{T_W\}(1-\rho)$$

We know:

$$\Rightarrow E\{T_W\} = \cancel{\frac{E\{T_R\}}{(1-\rho)}} \frac{\lambda \cdot E\{T_S^2\}}{2(1-\rho)}$$

$$\Rightarrow E\{T_R\} = \frac{\lambda \cdot E\{T_S^2\}}{2\cancel{(1-\rho)}} \cdot \cancel{(1-\rho)}$$

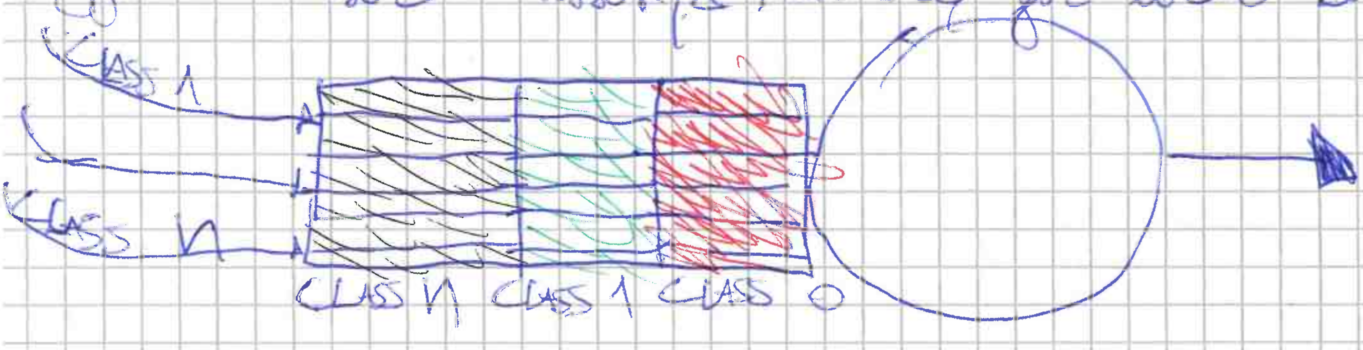
RESIDUAL TIME

$$\Rightarrow \boxed{E\{T_R\} = \frac{\lambda \cdot E\{T_S^2\}}{2}}$$

UNINTERRUPTABLE RESIDUAL TIME!

b) M/G/1 QUEUE WITH PRIORITIES for USER CLASSES — VIRTUAL & RESIDUAL TIME

There exist different waiting lines for the different user classes (& PRIORITY for each one).



PRIORITY INDEX = Inversely proportional to the PRIORITY itself

↳ [0 = TOP PRIORITY
n = LOWEST PRIORITY] ⇒ QoS, deterministic BEHAVIOUR

STARVATION in PACKET SWITCHING is BAD!
(Ex: If GOLD packets keep arriving & BRONZE packets are waiting in line, STARVATION of BRONZE packets would occur).

Now, we are interested in determining $E\{T_{W_i}\}$ for the i-th user class [Now also considering PRIORITIES!]

$$\boxed{E\{T_{W_i}\} = \sum_{j=1}^{i} E\{T_j\} + \sum_{j=1}^{i-1} E\{T_j'\} + E\{T_R\}}$$

$E\{T_V\}$ Because (j) is the user class.

USER CLASSES

Where: $\sum_{j=1}^{i} E\{T_j\}$ = TIME to wait for customers already in the waiting line to be SERVED.

$\sum_{j=1}^{i-1} E\{T_j'\}$ = TIME to wait for customers with priority lower than mine to be SERVED, having "surpassing" me in the waiting line
$\Rightarrow$ Customers with lower priority arriving while I wait.

$E\{T_R\}$ = RESIDUAL SERVICE TIME of the customer in the service center as I arrive to the queue

$$E\{T_j\} = \underbrace{E\{n_{W_j}\}}\cdot E\{\tau_{S_j}\}$$
$$= \lambda_j \cdot E\{T_{W_j}\} \cdot E\{\tau_{S_j}\}$$
$$= \rho_j \cdot E\{T_{W_j}\}$$

LOWER PRIORITY CLASS $\Downarrow$ Actually higher than other ones!

$$E\{T_j''\} = E\{n_{wj}''\} \cdot E\{T_{sj}\}$$

⇒ Customers in waiting line with priority lower than mine.

$$= \lambda_j \cdot \boxed{E\{T_{wi}\}} \cdot E\{T_{sj}\}$$

(1) waits

$$= \rho_j \cdot E\{T_{wi}\}$$

$$E\{T_R\} = \frac{\lambda}{2} E\{T_s^2\}$$

Because of NO PREEMPTION!

⇒ Putting it all together, we now have:

1 Priority ⇒ > (i) [actually lower!]

$$\boxed{E\{T_{wi}\}} = \sum_{j=1}^{i} E\{T_j'\} + \sum_{j=1}^{i-1} E\{T_j''\} + E\{T_R\}$$

$$= \sum_{j=1}^{\boxed{i}} \rho_j \cdot E\{T_{wj}\} + \sum_{j=1}^{i-1} \rho_j \, E\{T_{wi}\} + \frac{\lambda}{2} \cdot E\{T_s^2\}$$

$$= \sum_{\boxed{j=1}}^{\boxed{i-1}} \rho_j \cdot E\{T_{wj}\} + \boxed{\rho_i \cdot E\{T_{wi}\}} + \sum_{\boxed{j=1}}^{\boxed{i-1}} \rho_j \cdot E\{T_{wi}\} + \frac{\lambda}{2} \cdot E\{T_s^2\}$$

$$= \sum_{j=1}^{i-1} \rho_j \cdot E\{T_{wj}\} + \rho_i \cdot \underline{E\{T_{wi}\}} + \underline{E\{T_{wi}\}} \sum_{j=1}^{i-1} \rho_j + \frac{\lambda}{2} \cdot E\{T_s^2\}$$

$$E\{T_{wi}\} \cdot \left(1 - \sum_{j=1}^{i-1} \rho_j - \rho_i\right) = \sum_{j=1}^{i-1} \rho_j \cdot E\{T_{wj}\} + \frac{\lambda}{2} \cdot E\{T_s^2\}$$

$$\Rightarrow E\{T_{wi}\}\left(1 - \sum_{j=1}^{i} \rho_j\right) = \sum_{j=1}^{i-1} \rho_j \cdot E\{T_{wj}\} + \frac{\lambda}{2} \cdot E\{T_s^2\}$$

$$\Rightarrow \boxed{E\{T_{w_i}\} = \frac{\sum_{j=1}^{i-1} \rho_j \cdot E\{T_{w_j}\} + \frac{\lambda}{2} \cdot E\{T_s^2\}}{1 - \sum_{j=1}^{i} \rho_j}}$$

WAITING TIME WITH PRIORITY!

Now set:

$$\boxed{R_i = \sum_{j=1}^{i} \rho_j} \qquad \boxed{R_0 = 0}$$

For $i = 1$, $E\{T_{w_1}\} = ?$ [TOP PRIORITY, GOLD]

$$E\{T_{w_1}\} = \frac{\cancel{\sum_{j=1}^{0} \rho_j \cdot E\{T_{w_j}\}} + \frac{\lambda}{2} E\{T_s^2\}}{1 - \sum_{j=1}^{1} \rho_j}$$

$$\Rightarrow \boxed{E\{T_{w_1}\} = \frac{\frac{\lambda}{2} \cdot E\{T_s^2\}}{1 - \rho_1}}$$

For $i = 2$

$$E\{T_{w_2}\} = \frac{\overbrace{\sum_{j=1}^{1} \rho_j \cdot E\{T_{w_j}\}}^{\rho_1 \cdot E\{T_{w_1}\}} + \frac{\lambda}{2} \cdot E\{T_s^2\}}{1 - \sum_{j=1}^{2} \rho_j}$$

$$\Rightarrow E\{T_{w2}\} = \frac{\frac{\lambda}{2} \cdot E\{T_S^2\} + \rho_1 \boxed{E\{T_{w1}\}}}{1 - \rho_1 - \rho_2}$$

~~$E\{T_{w2}\}$~~ For $i = 3$:

$$E\{T_{w3}\} = \frac{\sum_{j=1}^{3} \rho_j \cdot E\{T_{wj}\} + \frac{\lambda}{2} \cdot E\{T_S^2\}}{1 - \rho_1 - \rho_2 - \rho_3}$$

$$E\{T_{w3}\} = \frac{\rho_1 \cdot E\{T_{w1}\} + \rho_2 \cdot E\{T_{w2}\} + \frac{\lambda}{2} E\{T_S^2\}}{1 - \rho_1 - \rho_2 - \rho_3}$$

Substitute $\boxed{E\{T_{w1}\}}$ into $\boxed{E\{T_{w2}\}}$

$$E\{T_{w2}\} = \frac{\rho_1 E\{T_{w1}\} + \frac{\lambda}{2} E\{T_S^2\}}{1 - \rho_1 - \rho_2}$$

$$\Rightarrow E\{T_{w2}\} = \frac{\rho_1 \frac{\frac{\lambda}{2} E\{T_S^2\}}{1 - \rho_1} + \frac{\lambda}{2} \cdot E\{T_S^2\}}{1 - \rho_1 - \rho_2}$$

$$= \frac{\frac{\lambda}{2} E\{T_S^2\} \cdot \left(\frac{\rho_1}{1 - \rho_1} + 1 \right)}{1 - \rho_1 - \rho_2}$$

$$= \frac{\frac{\lambda}{2} \cdot E\{T_S^2\} \cdot \left(\frac{\cancel{\rho_1} + 1 - \cancel{\rho_1}}{1 - \rho_1} \right)}{1 - \rho_1 - \rho_2}$$

$$\Rightarrow E\{T_{w2}\} = \frac{\frac{\lambda}{2} \cdot E\{T_S^2\}}{(1-\rho_1)\cdot(1-\rho_1-\rho_2)}$$

$\Rightarrow$ IN GENERAL:

$$\boxed{E\{T_{wi}\} = \frac{\lambda}{2} \cdot E\{T_S^2\} \cdot \frac{1}{(1-R_{i-1})\cdot(1-R_i)}}$$

WAITING TIME for the i-th-USER CLASS $M|G|1$ under PRIORITIES

$$\boxed{E\{T_{wi}\} = \frac{E\{T_R\}}{(1-R_{i-1})(1-R_i)}}$$

NB: WAITING TIME WITHOUT PRIORITY

$$E\{T_w\} = E\{T_{w1}\} \cancel{= \frac{E\{T_R\}}{(1-R_{i-1})\cdot(1-R_i)}}$$

$$\boxed{E\{T_w\} = \frac{\frac{\lambda}{2} \cdot E\{T_S^2\}}{(1-\rho)}}$$

$$\boxed{R_i = \sum_{j=1}^{i} \rho_j} \qquad \boxed{R_{i-1} = \sum_{j=1}^{i-1} \rho_j}$$

FORMULAS' SUMMARY: NO PRIORITY

PER CLASS:

$$E\{T_i\} = E\{T_{wi}\} + E\{\tau_{si}\}$$

$$E\{n_{wi}\} = \lambda_i \cdot E\{T_{wi}\}$$

$$E\{n_i\} = \lambda_i \cdot E\{T_i\} = \underbrace{\rho_i}_{E\{n_{si}\}} + E\{n_{wi}\}$$

GLOBAL:

$$E\{T_w\} = \frac{1}{\lambda} \cdot \sum_{i=1}^{n} \lambda_i \cdot E\{T_{wi}\}$$

NOT CUMULATIVE

$$E\{T\} = \frac{1}{\lambda} \sum_{i=1}^{n} \lambda_i \cdot E\{T_i\} = E\{\tau_s\} + E\{T_w\}$$

L.F.

$$E\{n_w\} = \sum_{i=1}^{n} E\{n_{wi}\} \overset{!}{=} \sum_{i=1}^{n} \lambda_i \cdot E\{T_{wi}\}$$

$$E\{n\} = \sum_{i=1}^{n} E\{n_i\} = \rho + E\{n_w\}$$

NB: $E\{T_{wi}\} = E\{T_w\}$ only if there is NO priority!

[i.e.: NO PRIORITY CLASSES]

$M/G/1$ QUEUE WITH MULTIPLE CLASSES: (PROOF ...)

T_{W} INVARIANCE TO USER CLASS considered!

$\boxed{E\{T_W\}}$ $E\{T_{W_i}\} = E\{T_W\}$

WITHOUT PRIORITIES:

$$E\{T_W\} = \frac{\frac{\lambda}{2} \cdot E\{T_S^2\}}{1-\rho}$$

WITH PRIORITIES:

$$E\{T_{W_i}\} = \frac{\frac{\lambda}{2} \cdot E\{T_S^2\}}{(1-R_{i-1})(1-R_i)}$$

$\boxed{E\{T_V\}}$

WITHOUT PRIORITIES (PRIORITY CLASSES)

$$E\{T_V\} = \frac{\lambda}{2} \cdot E\{T_S^2\} \cdot \frac{\rho}{1-\rho}$$

$$\Rightarrow \boxed{E\{T_V\} = \rho \cdot E\{T_W\}}$$

WITH PRIORITY NO PRIORITY

PROVE $E\{T_{V_i}\} = E\{T_V\}$

WITH PRIORITIES (PRIORITY CLASSES)

$$E\{T_V\} = \sum_{i=1}^{n} E\{N_{W_i}\} \cdot E\{T_{S_i}\}$$ [All customers in other user classes]

$$= \sum_{i=1}^{n} \lambda_i \cdot E\{T_{W_i}\} \cdot E\{T_{S_i}\}$$

$$= \sum_{i=1}^{n} \rho_i \cdot \boxed{E\{T_{W_i}\}}$$

Because it's M/G/1 with PRIORITY

$$= \cancel{\frac{\lambda}{2} \cdot E\{T_S^2\} \cdot \sum_{i=1}^{n} \frac{\rho_i}{(1-R_{i-1}) \cdot (1-R_i)}}$$

$$= \sum_{i=1}^{n} \rho_i \cdot E\{T_{W,i}\}$$

$$= \sum_{i=1}^{n} \rho_i \cdot \frac{\frac{\lambda}{2} \cdot E\{T_S^2\}}{(1-R_{i-1})(1-R_i)}$$

$$= \frac{\lambda}{2} \cdot E\{T_S^2\} \cdot \sum_{i=1}^{n} \frac{\rho_i}{\boxed{(1-R_{i-1})(1-R_i)}}$$

Consider $\boxed{(1-R_{i-1}) - (1-R_i)} = \rho_i$ $\quad R_i = \sum_{j=1}^{i} \rho_j \quad R_{i-1} = \sum_{j=1}^{i-1} \rho_j$

$$(1-R_{i-1}) - (1-R_i) = \left(1-\sum_{j=1}^{i-1} \rho_j\right) - \left(1-\sum_{j=1}^{i} \rho_j\right)$$

$$= \left(\cancel{1}-\cancel{\rho_1}-\cancel{\rho_2}-\cancel{\rho_3} \ldots - \rho_{i-1}\right) - \left(\cancel{1}-\cancel{\rho_1}-\cancel{\rho_2}-\ldots-\boxed{\rho_i}\right)$$

$\Rightarrow$ ρ_i is the only survivor.

$$\Rightarrow \boxed{(1-R_{i-1}) - (1-R_i) = \rho_i}$$

$$\Rightarrow E\{T_W\} = \frac{\lambda}{2} \cdot E\{T_S^2\} \cdot \sum_{i=1}^{n} \frac{(1-R_{i-1})-(1-R_i)}{(1-R_{i-1})(1-R_i)}$$

Remember: $R_0 = 0$ & BREAK the FRACTIONS:

$$\Rightarrow \frac{\lambda}{2} E\{T_S^2\} \left[\sum_{i=1}^{n} \frac{1}{1-R_i} - \sum_{i=1}^{n} \frac{1}{1-R_{i-1}}\right]$$

$$= \left(\cancel{\frac{1}{1-R_1}} + \cancel{\frac{1}{1-R_2}} + \ldots + \cancel{\frac{1}{1-R_{n-1}}} + \frac{1}{1-R_n}\right) -$$

$$\left(\frac{1}{1} + \cancel{\frac{1}{1-R_1}} + \cancel{\frac{1}{1-R_2}} + \ldots + \ldots \frac{1}{\ldots}\right)$$

$\Rightarrow$ 1 and $\dfrac{1}{1-R_n}$ are the only SURVIVORS.

$$\Rightarrow E\{T_V\} = \frac{\lambda}{2} \cdot E\{T_S^2\} \left[\frac{1}{1-R_n} - 1\right]$$

$$\Rightarrow E\{T_V\} = \frac{\lambda}{2} \cdot E\{T_S^2\} \cdot \left(\frac{\cancel{1} - \cancel{1} + R_n}{1 - R_n}\right)$$

$R_n = \rho$

$$\Rightarrow \boxed{E\{T_V\} = \frac{\lambda}{2} \cdot E\{T_S^2\} \cdot \frac{\rho}{1-\rho}} = \rho \cdot E\{T_W\}$$

VIRTUAL TIME ~~WITHOUT~~ PRIORITIES is INVARIANT to the USER CLASS considered!

& it is equal to the VIRTUAL TIME without PRIORITIES:

$$E\{T_V\} = \frac{\lambda}{2} E\{T_S^2\} \cdot \frac{\rho}{1-\rho} = \rho \cdot E\{T_W\}$$

~~$\Rightarrow$ WITH PRIORITY E~~

$E\{T_V\}$

WITHOUT PRIORITY:

$$\boxed{E\{T_V\} = \rho \cdot E\{T_W\}}$$

WITH PRIORITY

$$\boxed{E\{T_V\} = \sum_{i=1}^{n} \rho_i \cdot E\{T_{W_i}\}}$$

$$\Rightarrow E\{T_V\} = \rho \cdot E\{T_W\} = \sum_{i=1}^{n} \rho_i \cdot E\{T_{W_i}\}$$

C ⇒ CONSERVATION LAW:

$$\boxed{\underbrace{E\{T_V\}}_{\text{NO PRIORITY}} = E\{T_W\} = \underbrace{\frac{\Lambda}{\rho}\sum_{i=1}^{n}\rho_i \cdot E\{T_{Wi}\}}_{\text{WITH PRIORITY}}}$$

$E\{T_V\}$

⇒ If having some classes with lower priority than others, there will be other classes with decreased performance.

⇒ BALANCED COEFFICIENTS!

when introducing PRIORITY

↳ Worse service with higher priority

↳ Better $\boxed{\text{service}}$ with lower priority

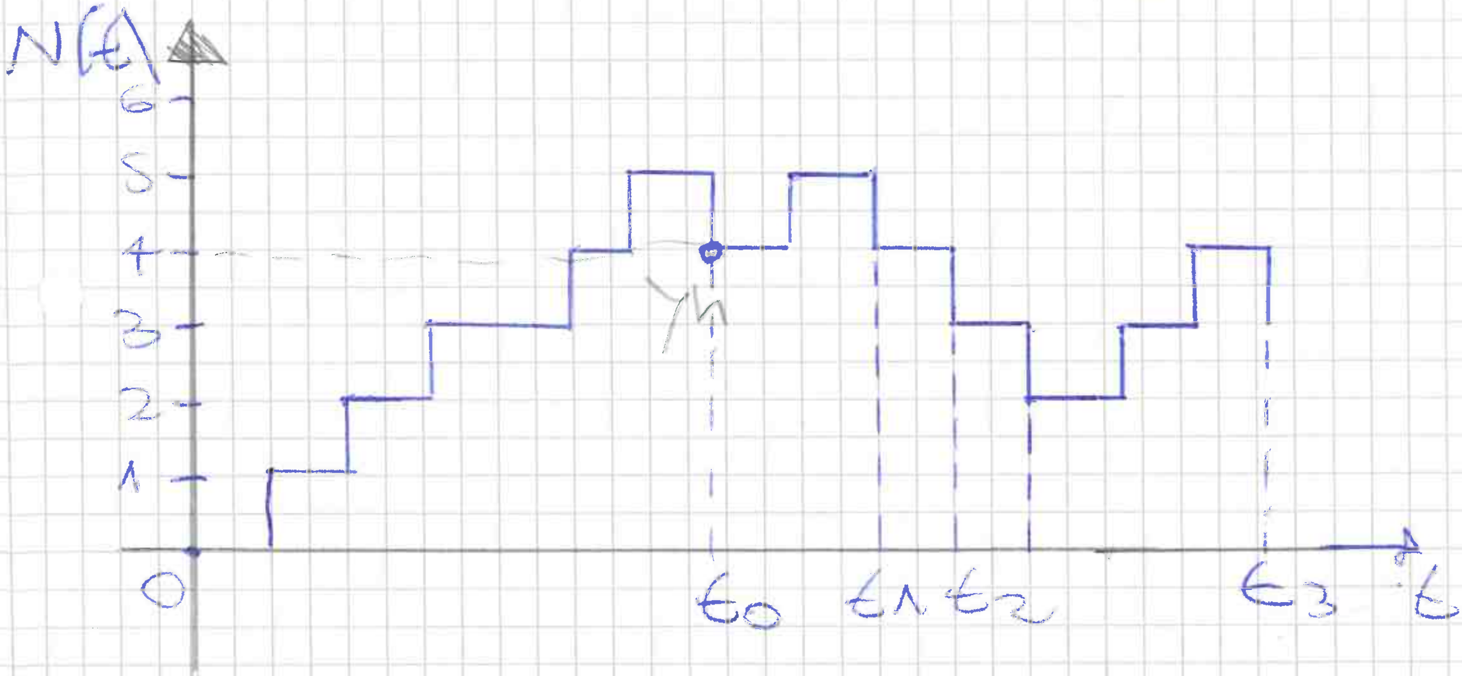
↳ WAITING TIME in line!

[~~Eventually~~ $\cancel{\sum_i \rho_i \lambda_i}$ ~~...~~ adding somewhere

⇒ Remaining somewhere!]

53) EMBEDDED MARKOV CHAIN APPROACH (DTMC) for M/G/1 QUEUES

We would like the STATE of the chain to be characterized uniquely by the # CUSTOMERS in the system



⇒ Now consider the following two R.Vs:

$\{Y_n\}$ = # CUSTOMERS left in the ~~system~~ QUEUE by the nth - DEPARTURE served

(~~the~~ CUSTOMERS I see when "turning ~~my~~" my back after leaving the system)

$\{Z_n\}$ = # CUSTOMERS arriving to the queue during the nth - SERVICE completed

customer going into the waiting line while I'm being

QUESTION: Could $\boxed{\{Y_N\}}$ be a $\boxed{DTMC}$?

PREVIOUS #CUSTOMERS, DEPARTURE

$$Y_{n+1} = \begin{cases} (1)\ \ Y_n + Z_{n+1} - 1 & Y_n > 0 \ \text{(QUEUE NOT empty)} \\ (2)\ \ 1 + Z_{n+1} - 1 & Y_n = 0 \ \text{(QUEUE is empty)} \end{cases}$$

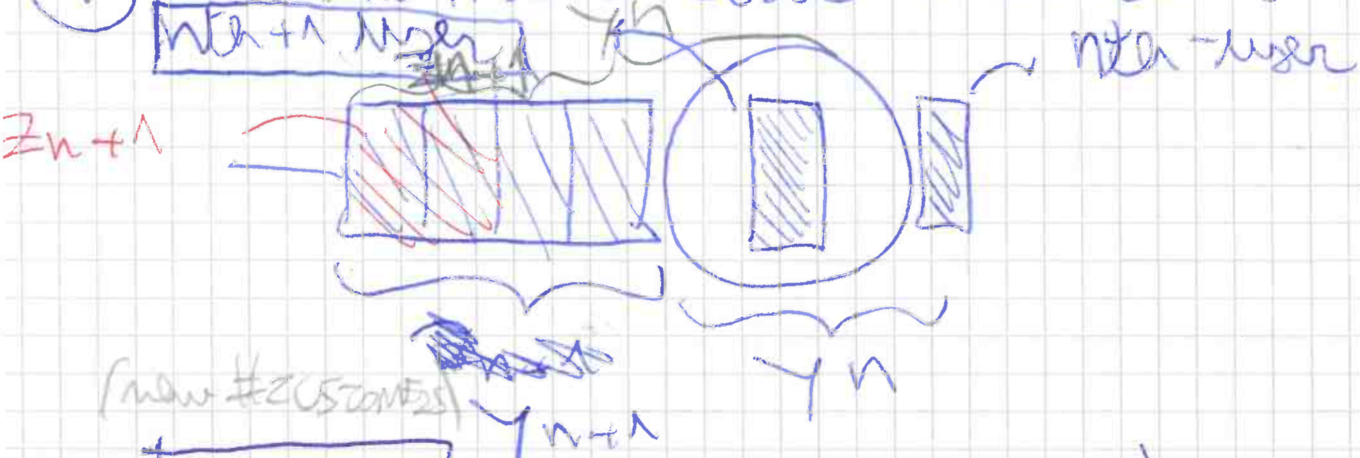
NEW CUSTOMERS ARRIVING

(n-1) 1 CUSTOMER ARRIVING

(n+1) CUSTOMERS DEPARTURE

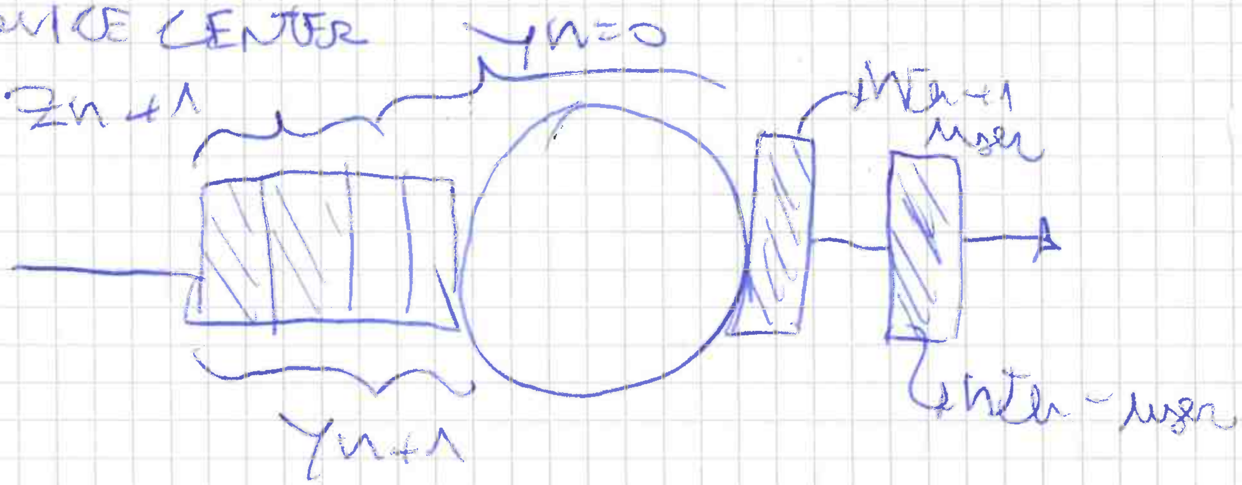
$\boxed{Y_n > 0}$: (WAITING LINE)

(1) Someone in the QUEUE & in SERVICE CENTER)



$\boxed{Y_n = 0}$ (WAITING LINE)

(2) Nobody in the QUEUE & nobody in the SERVICE CENTER



$\Rightarrow$ if we now define:

$$\mu(t) = \begin{cases} 1 & \text{if } Y_n(t) > 0 \\ 0 & \text{if } Y_n(t) = 0 \end{cases}$$

⇒ We can then express Y_{n+1} as:

$$\boxed{Y_{n+1} = Y_n + Z_{n+1} - (\mu(Y_n))}$$

And if Y_N were a Markov Chain, then we would have that:

$$h_{ij} = P\{Y_{n+1} = j \mid Y_n = i\}$$

"Is this true"? ⇒ Do we have a HDTMC?

~~⇒ Nothing~~

"If we look at certain time intervals of the M|G|1 QUEUE, do we have a $\boxed{DTMC}$?" ⇒ DEPT ONLY (customers take care)

↳ SPOILER: Yes, but only upon departure from the system.

NON-MARKOVIAN

⇒ A DTMC found in an M|G|1 QUEUE is known as a HIDDEN MARKOV CHAIN (EMBEDDED)

BY DO SUBSTITUTE VARIABLES

$$h_{ij} = P\{\boxed{Y_{n+1} = j \mid Y_n = i}\} \Rightarrow \text{CHECK if this holds true}$$

$$j = i + Z_{n+1} - \mu(Y_n).$$

$$h_{ij} = P\{Z_{n+1} - \mu(Y_n) = j - i\}$$

Where:

ONE-STEP TRANS. PROB MATRIX

$i = Y_N = 0$ [FIRST row of the matrix] ⇒ $P\{Z_{n+1} = j\} = P_j$ PMF

$i = Y_N > 0$ [NEXT rows of the matrix] ⇒ $P\{Z_{n+1} = j - i + 1\} = P_{j-i+1}$

⇒ We now want to find:

$\boxed{P\{Z_n = k\}}$ ⇒ Apply TOT. PROB. THEOREM in CONTINUOUS CASE

Z_n ↳ # CUSTOMERS arriving during the ~~(n-1)~~ departure SERVICE

$$P\{Z_n=k\} = \int_0 P\{Z_n=k \mid S=\theta\}\, f_S(\theta)\, d\theta$$

ALL CONTRIBUTIONS ARE ...

f_S ↳ PDF of hence CHOSEN

By HOMOGENEITY

⇒ POISSONIAN PROCESS in θ ⇒ POISSONIAN ARRIVALS (like a MARKOFF CHAIN)

$$= \int_0^{+\infty} \frac{\lambda\theta}{k!}\, e^{-\lambda\theta}\, f_S(\theta)\, d\theta$$

$$\boxed{P\{Z_n=k\} = P_K}$$

[Provided that QUEUE is ERGODIC / STABLE

$$\lambda \cdot E\{S\} < 1$$

$\boxed{A < 1}$ i.e.: All the offered traffic is ACCEPTED

We here have a
⇒ HIDDEN MARKOV CHAIN
⇒ We now write the MATRIX of TRANSITION PROBABILITIES:

i-th row

$$\underline{\underline{H}} = \begin{bmatrix} p_0 & p_1 & p_2 & \dots & p_K \\ p_0 & p_1 & p_2 & \dots & p_K \\ 0 & p_0 & p_1 & \dots & p_{K-1} \\ 0 & 0 & p_0 & p_1 \dots & p_{K-2} \end{bmatrix}$$

$h_{ij} = p_j$ if $i=0$ ∀N

$h_{ij} = p_{j-i+1}$ if $i \geq 0$ ∀N

Thanks to p_K → HIDDEN MARKOV CHAIN

⇒ It is a HOMOGENEOUS DTMC and we ~~[illegible]~~ can evaluate its STEADY-STATE PROBABILITIES

$$\pi^d \cdot \underline{\underline{\mu}} = \pi^d$$ BIASED AT EXIT (in ERGODIC CONDITIONS)

(STEADY-STATE PROBABILITY) — UNITARY VARIATIONS ONLY (PASTA) → STEADY-STATE EXACTLY ⇓ Evaluate the STATIONARY PROBABILITY

$$\Rightarrow \boxed{\pi_j^{(DEP.)} = \pi_j^{(ARR)} = \pi_j^d}$$

what you see at the DEPARTURE = what you see at the ARRIVAL (what a random customer sees), while where

$$\boxed{\pi_j^d = P\{Y_n = j\}}$$

⇒ We hence have a HDTMC upon CUSTOMER DEPARTURE (i.e.: the state is only given by the # CUSTOMERS in the system AT CUSTOMER DEPARTURE)

(S4) MEAN VALUE ANALYSIS — PROOF OF P-K FORMULA

(on the way to P-K FORMULA)

Recall:
$$\boxed{Y_{n+1} = Y_n + Z_{n+1} - \mu(Y_n)}$$

$E\{\cdot\}$ of all terms:
$$\Rightarrow E\{Y_{n+1}\} = E\{Y_n\} + E\{Z_{n+1}\} - E\{\mu(Y_n)\}$$

with $\boxed{\lambda \cdot E\{S\} < 1}$ (ERGODICITY CONDITION)

Because of ERGODICITY, a STEADY-STATE exists:

$$\lim_{n \to \infty} E\{Y_{n+1}\} = E\{Y_n\} = E\{Y\}$$

$$\lim_{n \to \infty} E\{\mu(Y_n)\} = E\{\mu(Y_n)\}$$

$$\lim_{n \to \infty} E\{Z_n\} = E\{Z_{n+1}\} = E\{Z\}$$

$$\Rightarrow E\{Y_{n+1}\} = E\{Y_N\} + E\{Z_{n+1}\} - E\{\mu(Y_n)\}$$

For $\lim_{n \to \infty}$: $E\{Y_N\} \to E\{Y\}$, $E\{Y\}$, $E\{Z\}$

$$\Rightarrow E\{Y\} = E\{Y\} + E\{Z\} - E\{\mu(Y_N)\}$$

$$\Rightarrow \boxed{E\{\mu(Y_N)\} = E\{Z\}}$$

$$\mu(Y) = \begin{cases} 0 & \text{if the queue is empty } Y_N = 0 \\ 1 & \text{if the queue is NOT empty } Y_N \neq 0 \end{cases}$$

$$\Rightarrow \boxed{E\{\mu(Y)\}} = \frac{E\{N_S\}}{N_S} \underset{M/G/1}{=} \frac{E\{N_S\}}{1} = \boxed{\rho}$$

$$\Rightarrow \boxed{E\{Z\} = \rho}$$

P-K FORMULA PROOF

GOAL: Find $\boxed{E\{Y\}}$

We know that:

$$\boxed{Y_{n+1} = Y_n + Z_{n+1} - \mu(Y_n)}$$

Square all terms!

$$\Rightarrow \cancel{(Y_{n+1})^2 = Y_n^2 + Z_{n+1}}$$

SQUARE the LEFT & RIGHT-HAND SIDE:

$$(y_{n+1})^2 = (y_n)^2 + (z_{n+1})^2 + \mu(y_n) + 2y_n \cdot z_{n+1} - 2y_n \cdot \mu(y_n) - 2z_{n+1} \cdot \mu(y_n)$$

$\mu(y_n) = y_n$ & $E\{\mu(y_n)\} = E\{z\}$

Take $E\{\cdot\}$ of all terms

$\boxed{z_{n+1}, y_n \text{ \& } z_{n+1}, \mu(y_n) \text{ INDEPENDENT}}$

$$\cancel{E\{y_{n+1}^2\}} = \cancel{E\{y_n^2\}} + \underbrace{E\{z_{n+1}^2\}}_{E\{z^2\}} + \underbrace{E\{\mu(y_n)\}}_{E\{z\}} + 2\underbrace{E\{y_n\}}_{E\{y\}} \cdot \underbrace{E\{z_{n+1}\}}_{E\{z\}} - 2\underbrace{E\{y_n\}}_{E\{y\}} \cdot E\{\mu(y_n)\} - 2\underbrace{E\{z_{n+1}\}}_{E\{z\}} \underbrace{E\{\mu(y_n)\}}_{E\{z\}}$$

$E\{\mu(y_n)\} = 1$

$$0 = E\{z^2\} + E\{z\} + 2E\{y\} \cdot E\{z\} - 2E\{y\} - 2(E\{z\})^2$$

We are interested in finding $E\{y\}$

$$-2E\{y\} \cdot E\{z\} + 2E\{y\} = E\{z^2\} + E\{z\} - 2(E\{z\})^2$$

$$2E\{y\} \cdot (1 - E\{z\}) = E\{z^2\} + E\{z\} \underset{\text{NEW}}{+E\{z\}} - E\{z\} - 2E^2\{z\}$$

$$2E\{y\} \cdot (1 - E\{z\}) = E\{z^2\} + \underbrace{2E\{z\} - 2E^2\{z\}}_{2E\{z\}[1 - E\{z\}]} - E\{z\}$$

$$\Rightarrow 2E\{y\}(1 - E\{z\}) = E\{z^2\} - E\{z\} + 2E\{z\}[1 - E\{z\}]$$

$$\Rightarrow E\{y\} = \frac{E\{z^2\} - E\{z\}}{2 \cdot (1 - E\{z\})} + \frac{\cancel{2}E\{z\}\cancel{[1 - E\{z\}]}}{\cancel{2}\cancel{[1 - E\{z\}]}}$$

$$\Rightarrow E\{Y\} = \underbrace{E\{Z\}}_{\rho} + \frac{E\{Z^2\} - E\{Z\}}{2\cdot[1 - E\{Z\}]}$$

$$\Rightarrow E\{Y\} = \rho + \frac{\boxed{E\{Z^2\}} - \rho}{2\cdot(1-\rho)}$$

Where $\boxed{E\{Z^2\}}$ is:

$$E\{Z^2\} = \sum_{k=1}^{+\infty} k^2 \cdot \overbrace{P\{Z=k\}}^{\text{POISSON-DISTRIBUTED}} = \sum_{k=1}^{+\infty} k^2 \cdot \int \frac{(\lambda\theta)^k}{k!} \cdot e^{-\lambda\theta} \cdot f_S(\theta)\, d\theta$$

$$= \int_0^{+\infty} \underbrace{\sum_{k=0,1}^{+\infty} k^2 \cdot \frac{(\boxed{\lambda\theta})^k}{k!} \cdot e^{-\lambda\theta}}_{=E\{Z^2\}} f_S(\theta)\, d\theta$$

Consider the POISSON DISTRIBUTION:

$E\{X\} \Rightarrow$ The AVERAGE # ARRIVALS in $(0,\theta) = \boxed{\lambda\theta}$

$VAR\{X\} \Rightarrow$ The VARIANCE is also $\boxed{\lambda\theta}$

$\Rightarrow$ Take the VARIANCE FORMULA: $E^2\{x\} = (E\{x\})^2$

$$VAR\{Z\} = E\{Z^2\} - E^2\{Z\}$$

$$E\{Z^2\} = E^2\{Z\} + VAR\{Z\}$$

$$E\{Z^2\} = \lambda^2\theta^2 + \lambda\theta$$

$$\Rightarrow E\{Z^2\} = \int_0^{+\infty} (\lambda^2\theta^2 + \lambda\theta) \cdot f_S(\theta)\, d\theta$$

BREAK IT!

$$= \lambda^2 \int_0^{+\infty} \theta^2 \cdot f_S(\theta)\, d\theta + \lambda \int_0^{+\infty} \theta \cdot f_S(\theta)\, d\theta$$

$$\Rightarrow \lambda^2 \int_0^{+\infty} \theta^2 \cdot f_S(\theta)\,d\theta + \lambda \int_0^{+\infty} \theta\, f_S(\theta)\,d\theta$$

$$= \lambda^2 \cdot \underbrace{E\{S^2\}}_{s^2} + \lambda \cdot \underbrace{E\{S\}}_{s}$$

$E\{z\}\rho = E\{S\}\lambda = E\{z\}$; $\lambda S = 1$

$$\Rightarrow E\{n\} = E\{Y\} = \rho + \frac{\lambda^2 \cdot E\{S^2\} + \cancel{\lambda E\{S\}} \cdot \rho}{2 \cdot (1-\rho) \cdot E\{z\}}$$

$$\Rightarrow \boxed{E\{n\} = \rho + \frac{\lambda^2 \cdot E\{S^2\}}{2(1-\rho)}}$$

P-K FORMULA, provided that:

$\lambda \cdot E\{S\} < 1$

$$C_S = \frac{\theta_S}{E\{S\}} \Rightarrow C_S^2 = \frac{\overbrace{\theta_S^2}^{\text{VARIANCE}}}{E^2\{S\}} = \frac{E\{S^2\} - E^2\{S\}}{E^2\{S\}}$$

$$E^2\{S\} \cdot C_S^2 = E\{S^2\} - E^2\{S\}$$

$$\Rightarrow \boxed{E^2\{S\} \cdot (1 + C_S^2) = E\{S^2\}}$$

$$E\{n\} = \rho + \frac{\lambda^2 \cdot E^2\{S\} \cdot (1 + C_S^2)}{2(1-\rho)}$$

If the service is EXPONENTIAL (i.e: like in M/M/1 QUEUES)

$C_S = 1$

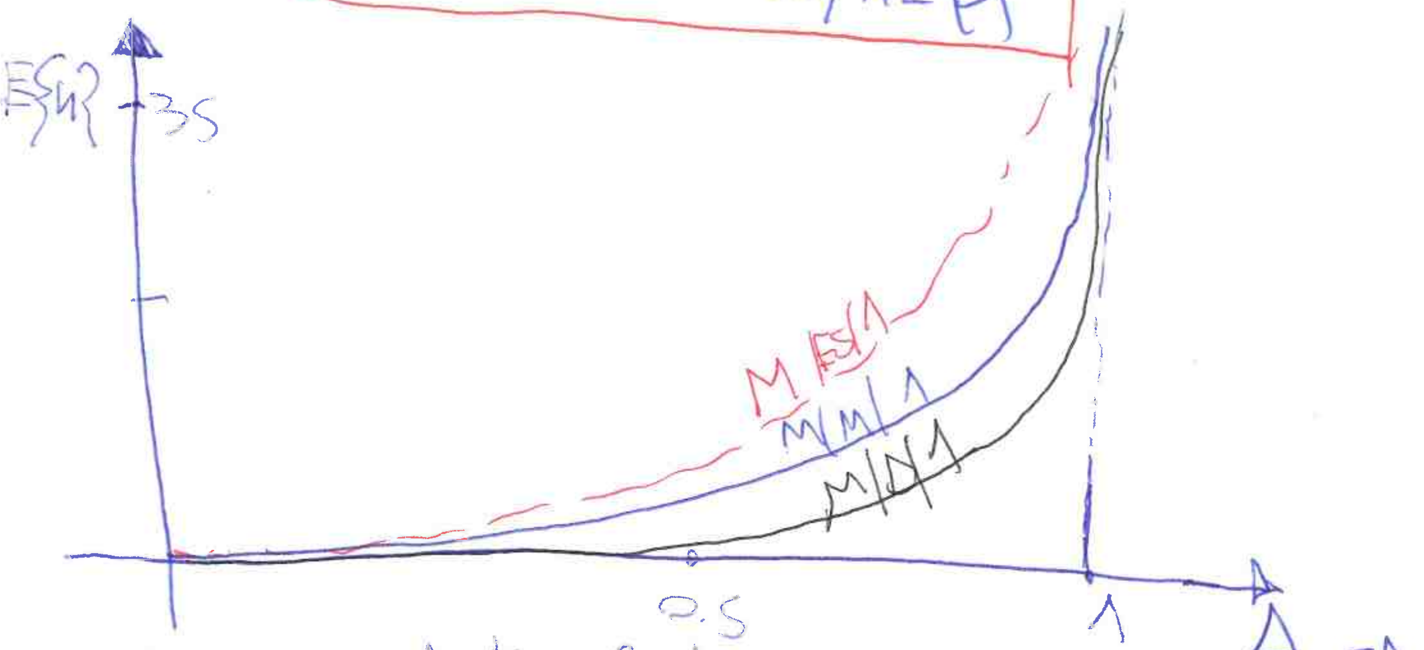
$$\Rightarrow E\{n\} = \rho + \frac{\rho^2}{1-\rho} = \frac{\rho(1-\rho) + \rho^2}{(1-\rho)} = \frac{\rho}{1-\rho}$$

AVERAGE QUEUEING TIMES IN M/G/1 QUEUES:

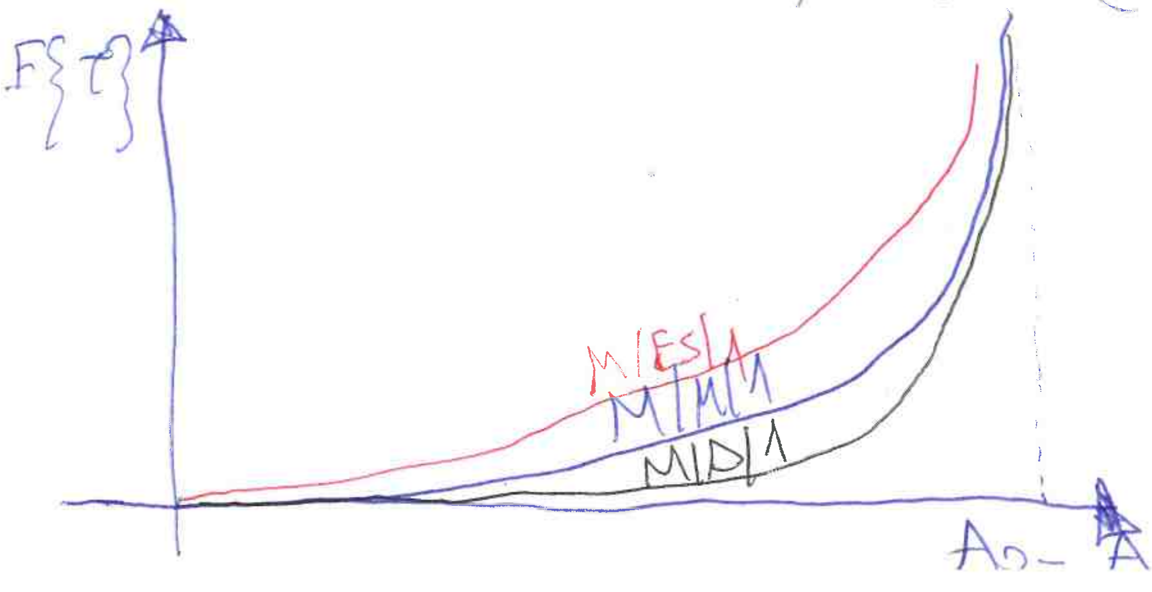
$$E\{\tau\} = \frac{E\{n\}}{\Lambda} = \frac{E\{n\}}{\lambda}$$

$$E\{\tau\} = E\{s\} + \frac{\lambda \cdot E\{s^2\}}{2\cdot(1-\rho)} = \underbrace{E\{s\}}_{E\{\tau_s\}} + \underbrace{\frac{\lambda E^2\{s\}(1+C}{2(1-\rho)}}_{E\{\tau_w\}}$$

$$\Rightarrow \boxed{E\{\tau_w\} = \frac{\lambda \cdot E\{s^2\}}{2\cdot(1-\rho)}}$$



M/D/1 provides the least queueing time.



LECTURE N6

2) PERCENTILES in M|M|1 – M|M|1|Nw QUEUE

Do not limit ourselves to the analysis of mean values, but consider the DISTRIBUTION as well ⇒ through distribution can get all MOMENTUMS.

PERCENTILES IN QUEUES

Especially useful in DATA-driven analysis of
⇒ Similar approach to CONFIDENCE INTERVALS

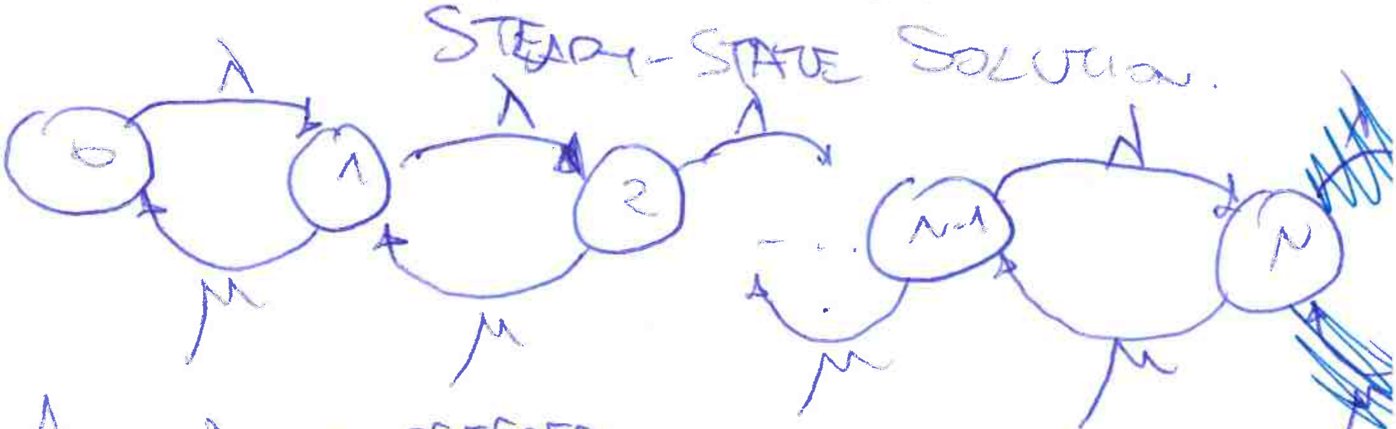
⇒ Consider M/M/1/NW QUEUE

N = Nw + 1 [FINITE-SIZE queue].

⇓

FINITE # STATES

⇒ If APERIODIC ⇒ Also have a STEADY-STATE SOLUTION.



$A_0 = \frac{\lambda}{\mu}$ [OFFERED TRAFFIC INTENSITY] ≠ A [ACCEPTED TRAFFIC INTENSITY]

~~$\sum_{k=0}^{N} P_{k} \lambda$~~

Find pn:

$$\lambda p_0 = \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0$$

$$\lambda p_1 = \mu p_2 \Rightarrow p_2 = \frac{\lambda}{\mu} p_1 = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu} \cdot p_0 = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

$$\lambda p_2 = \mu p_3 \Rightarrow p_3 = \frac{\lambda}{\mu} p_2 = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu} \frac{\lambda}{\mu} p_0 = \left(\frac{\lambda}{\mu}\right)^3 p_0$$

$$\Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot p_0$$

$$A_0 = \frac{\lambda}{\mu} \Rightarrow \boxed{p_n = (A_0)^n \cdot p_0}$$

$\Rightarrow$ Find p_0 by applying $\boxed{\sum_{k=0}^{N} p_k = 1}$

$$\sum_{n=0}^{N} (A_0)^n \cdot p_0 = 1$$

$0 \leq n \leq N$

$$\Rightarrow \boxed{p_0 = \frac{1}{\sum_{n=0}^{N} (A_0)^n}} = \boxed{\frac{1 - A_0}{1 - A_0^{N+1}}}$$

$$\boxed{\sum_{i=0}^{N} (a)^i = \frac{1 - a^{N+1}}{1 - a}}$$

$\Rightarrow$ We can then substitute p_0 into p_n:

$$\boxed{p_n = (A_0)^n \cdot \frac{1 - A_0}{1 - A_0^{N+1}}}$$

Presently, we are interested in:

$$\boxed{P_L = P_B}$$ (because exogenous process).

$$P_L = P_B = A_0^N \cdot \frac{(1-A_0)}{(1-A_0^{N+1})}$$

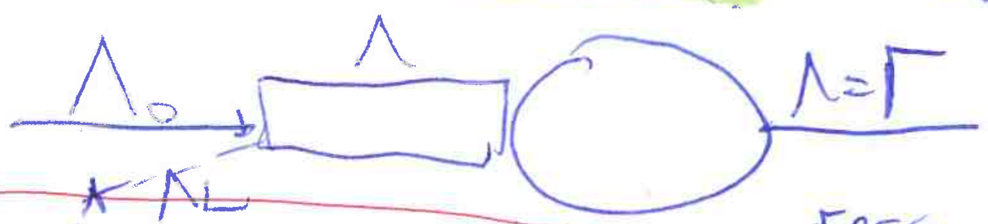
$$E\{ns\} = \boxed{A = A_0(1-P_L)}$$

ACCEPTED ~~OFFERED~~ TRAFFIC INTENSITY (AVG. # BUSY SERVERS)

$$\boxed{A_L = P_L \cdot A_0}$$

REJECTED TRAFFIC INTENSITY (when saturated queue)

~~TO~~ THROUGHPUT ⇒ INTENSITY of exiting customers from the queue.



$$\boxed{\Gamma = \Lambda = \Lambda_0(1-P_L)} \Rightarrow \Gamma$$ FREQUENCY ~~INTENSITY~~ of EXITING CUSTOMERS

FREQUENCY ~~INTENSITY~~ of ACCEPTED CUSTOMERS

$$\boxed{\Lambda_L = \Lambda_0 \cdot P_L}$$

↳ FREQUENCY of REJECTED (lost) CUSTOMERS

$$P_n = A_0^{\,n} \cdot \frac{1 - A_0}{1 - A_0^{N+1}} \qquad 0 \leq n \leq N$$

$$\Rightarrow E\{n\} = \sum_{k=0}^{N} n\, p_n = \frac{1 - A_0}{1 - A_0^{N+1}} \sum_{n=0}^{N} n\, A_0^{\,n}$$

$$= \frac{1-A_0}{1-A_0^{N+1}} \sum_{n=0}^{N} n \cdot A_0^{\,n-1} \cdot \cancel{A_0}$$

NB considering BLOCKING CASE

(① way: Use known series' formula)

② way: Make the DERIVATIVE

$$\frac{1-A_0}{1-A_0^{N+1}} \sum_{n=0}^{N} n \cdot A_0^{\,n-1} = \boxed{\frac{d}{dA_0} \frac{1-A_0^{N+1}}{1-A_0}}$$

$$\cancel{\Rightarrow E\{n\} = \frac{A_0}{1-A_0} \cdot (1+N)}$$

$$= \frac{\left[(1-A_0)^{N+1}\right]' \cdot (1-A_0) - (1-A_0^{N+1}) \cdot (1-A_0)'}{(1-A_0) \cdot (1-A_0)^{N+1}}$$

$$\boxed{E\{n\} = \frac{A_0}{(1-A_0)} \; \frac{1 + N \cdot A^{N+1} - (N+1) \cdot A_0^{\,N}}{1 - A_0^{N+1}}}$$

$$\Rightarrow E\{T\} = \frac{E\{n\}}{\Lambda} \qquad \text{where } \Lambda = \Gamma = \lambda(1-P_L)$$

$$\sum_{n=0}^{N} n \cdot A^{n-1} = \frac{d}{dA_0} \frac{1 - A_0^{N+1}}{1 - A_0}$$

$$\Lambda_L = \lambda \cdot P_L$$

$$E\{n\} = \frac{A_0}{1-A_0} \cdot \frac{1 + N \cdot A_0^{N+1} - (N+1) \cdot A_0^{N}}{1 - A_0^{N+1}}$$

$$\Rightarrow \boxed{E\{T\} = \frac{E\{n\}}{\Lambda}}$$

SO FAR, we've only considered mean values. NOW:

→ Not only interested in mean value, but also in the distribution of the values

→ EX: VOICE DELAY ≤ 150 ms ←

(MAXIMUM is relevant too for QoS

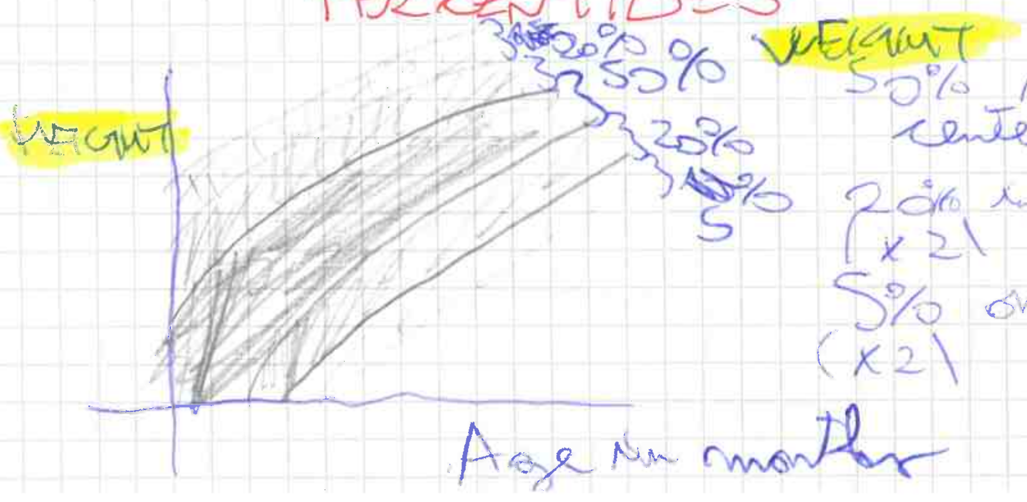
⇒ DETERMINISTIC STRICT GUARANTEE!

→ WORST-CASE ANALYSIS → NETWORK CALCULUS

Theory for crossing loops (Book)

Performance of packet-switched networks with QoS & scheduling.

PERCENTILES



50% median in the center

20% in-between (x2)

5% on the extremes (x2)

PERCENTILE [Considered ~~...~~ probability out of 1-100]

Value that does not surpass a certain threshold with a certain ~~threshold~~ ~~probability~~ [ε] VALUE

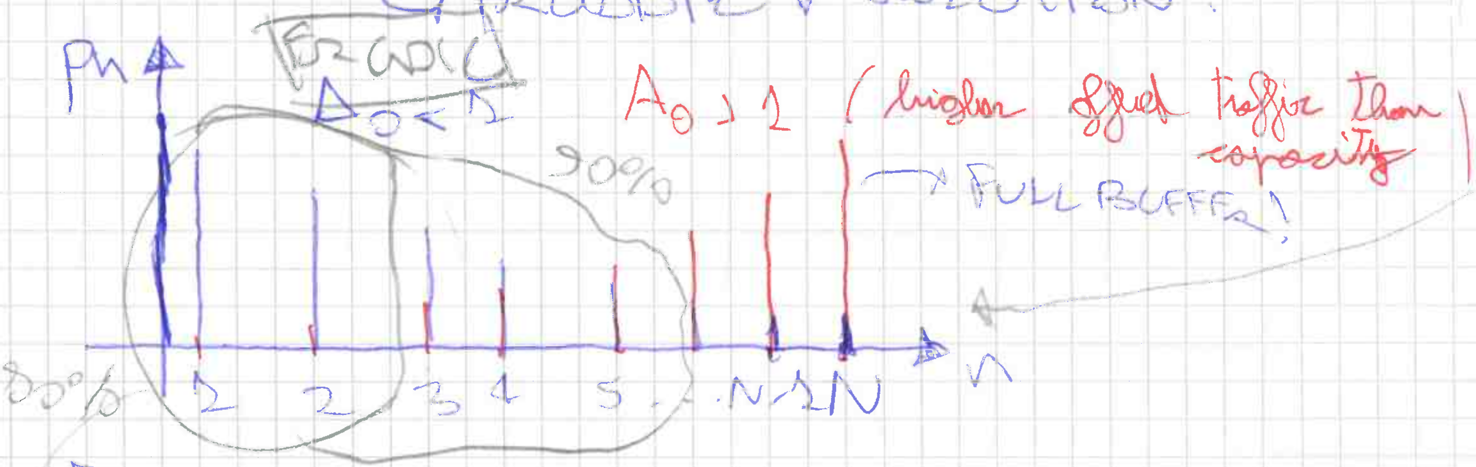
$$\boxed{P\{n \leq N_\varepsilon\} = \varepsilon}$$

ε → PERCENTAGE / PROBABILITY

N_ε → VALUE (THRESHOLD)

EX:
50% PERCENTILE → Value (not being surpassed with 50% probability)

For an M|M|1|N W
↳ ERGODIC SOLUTION!



Ex: Aloha → Issues with BUFFER

Ig $\varepsilon = 0.8 \Rightarrow N_{0.8} = 2$ (80%)

Ig $\varepsilon = 0.9 \Rightarrow N_{0.9} = 6$ (90%)

Discrete DISTRIBUTION

DECILES [10-20-30 - . . . - 100]

1ST DECILE ⇒ 10% PERCENTILE

2ND DECILE ⇒ 20% PERCENTILE

QUARTILES

1ST QUARTILE ⇒ 25% [0-25] 2ND QUARTILE [25-50%]

8) PERCENTILE FOR M|M|1|N QUEUES

d) FINDING PERCENTILE'S VALUE (for M/M/1/N)

I want to find this

$$\boxed{P\{n \leq \boxed{N_\varepsilon}\} = \sum_{n=0}^{N_\varepsilon} p_n = \varepsilon} \Rightarrow \text{PERCENTILE'S VALUE}$$

GOAL:

Find N_ε

Sum all bins of the distribution up to the desired VALUE (N_ε)

We know:

$$\boxed{P_n = (A_0)^n \cdot \frac{(1-A_0)}{1-A_0^{N+1}}}$$

$$\Rightarrow \varepsilon = \frac{(1-A_0)}{1-A_0^{N+1}} \sum_{n=0}^{N_\varepsilon} A_0^n$$

GOAL: Find N_ε

$$= \frac{\cancel{(1-A_0)}}{1-A_0^{N+1}} \cdot \frac{1-A_0^{N_\varepsilon+1}}{\cancel{1-A_0}}$$

$$\varepsilon = \frac{1-A_0^{N_\varepsilon+1}}{1-A_0^{N+1}}$$

$$\Rightarrow \varepsilon(1-A_0^{N+1}) = 1-A_0^{N_\varepsilon+1}$$

$$\Rightarrow A_0^{N_\varepsilon+1} = 1-\varepsilon(1-A_0^{N+1})$$

~~$\log(A_0^{N_\varepsilon+1}) = \log[1-\varepsilon(1-A_0^{N+1})]$~~

~~$\log(A_0^{N_\varepsilon}) + \log(A_0^1) = \log[1-\varepsilon(1-A_0^{N+1})]$~~

$$\Rightarrow 1 + N_\varepsilon = \frac{\log\left[1-\varepsilon(1-A_0)^{N+1}\right]}{\log A_0}$$

$$\Rightarrow N_\varepsilon = \frac{\log\left[1-\varepsilon(1-A_0)^{N+1}\right]}{\log A_0} - 1$$

By the LOGARITHMS' RULES:

$$\boxed{\log_b(M^k) = k \cdot \log_b M}$$

$$\log\left(A_0^{N_\varepsilon+1}\right) = \log\left[1-\varepsilon\left(1-A_0^{N+1}\right)\right]$$

$$(N_\varepsilon+1)\log(A_0) = \log\left[1-\varepsilon\left(1-A_0^{N+1}\right)\right]$$

$$\Rightarrow N_\varepsilon + 1 = \frac{\log\left[1-\varepsilon\left(1-A_0^{N+1}\right)\right]}{\log A_0}$$

$$\Rightarrow \boxed{N_\varepsilon = \frac{\log\left[1-\varepsilon\left(1-A_0^{N+1}\right)\right]}{\log A_0} - 1}$$

$0 \le \varepsilon \le 1 \Rightarrow$ Not-integer N_ε
→ Write a proper ε, can find an DISCRETE ε

b) FINDING PERCENTILE VALUE FOR M|M|1|∞ QUEUE:

We know: $\boxed{P_n = A_0^n \cdot (1-A_0)}$ → $P_n = (1-\rho)\,\rho^n$

$A_0 < 1$

$$\varepsilon = \sum_{n=0}^{N_\varepsilon} P_n = (1-A_0)\sum_{n=0}^{N_\varepsilon} A_0^n$$

$$\Rightarrow \varepsilon = \cancel{(1-A_0)} \cdot \frac{\left(1-A_0^{N_\varepsilon+1}\right)}{\cancel{(1-A_0)}}$$

Again, we are interested in N_ε.

$$\varepsilon = 1 - A_0^{N_\varepsilon + 1}$$

$$\Rightarrow \boxed{A_0^{N_\varepsilon+1} = 1 - \varepsilon}$$

$$\log(A_0^{N_\varepsilon+1}) = 1 - \varepsilon$$

$$\Rightarrow (N_\varepsilon + 1) \cdot \log(A_0) = \log(1-\varepsilon)$$

$$\Rightarrow \boxed{N_\varepsilon = \frac{\log(1-\varepsilon)}{\log(A_0)} - 1}$$ for $M|M|1|\infty$

Where if $N_\varepsilon \approx \varepsilon \cdot A_0^{N+1} << 1 - \varepsilon$

$\Rightarrow M|M|1 \approx M|M|1|N_W$

(Similar approximation of the $\underbrace{\text{INFINITE BUFFER}}_{N_W=\infty}$ with $\underbrace{\text{FINITE BUFFER}}_{N_W = N_W}$)

If $N_\varepsilon \approx \varepsilon \cdot A_0^{N+1}$

$\Rightarrow M|M|1|N_W$ is a GOOD APPROXIMATION of an $M|M|1|\infty$ QUEUE

59

PDF of WAITING TIME in M|M|1 QUEUES

$E\{T_W\}$

We know that in steady state: EXP. DISTRIBUTED

$$\boxed{p_n = (1-\rho)\cdot\rho^n}, \quad \rho = \frac{\lambda}{\mu}$$

We would like to find:

$$\boxed{f_{T_W}(t)} \Rightarrow \text{PDF of the WAITING TIME.}$$

IDEA.

Find $f_{T_W}(t|n)$, i.e.: PDF of waiting time based on the #CUSTOMERS you see upon arriving to the queue.

$$\boxed{f_{T_W}(t|n) = \begin{cases} \delta(t) & n=0 \\ \frac{\mu(\mu t)^{n-1}}{(n-1)!}\cdot e^{-\mu t} & n \geq 1 \end{cases}}$$

PDF of R.V. that has a deterministic... NO CUSTOMERS in waiting line

1 CUSTOMER: ⇒ Erlang-1 = Exponential R.V.

Exponential service ⇒ Wait 2 (whole SERVICE) RESIDUAL EXPONENTIAL

2 CUSTOMERS:

~~Res~~ Wait: $\underbrace{\text{RESIDUAL}}_{\text{exponential service}}$ + Exponential SERVICE

⇒ Erlang-2 R.V.

IN GENERAL

$$\boxed{(n) \text{ CUSTOMERS} \Rightarrow \text{Erlang-}n \text{ R.V.}}$$

~~⇒ In general~~

⇒ We can put everything together to find $f_{T_W}(t)$ (considering all possible (CONDITIONED) cases).

$$\Rightarrow f_{T_W}(t) = \sum_{n=0} f_{T_W}(t|n) \cdot p_n \text{ (arr)}$$

→ WEIGHT

PASTA

because POISSON ARRIVALS ⇒ [Probabilities at entrance are same as random arrivals]

$$= \sum_{n=0}^{+\infty} f_{T_W}(t|n) \cdot p_n$$

$$\Rightarrow f_{T_W}(t) = \underbrace{(1-\rho)\cdot\delta(t)}_{n=0} + \underbrace{\sum_{n=1}^{+\infty} (1-\rho)\cdot\rho^n \cdot \frac{\mu(\mu t)^{n-1}\cdot e^{-\mu t}}{(n-1)!}}_{n\geq 1}$$

$$= (1-\rho)\cdot\delta(t) + \mu(1-\rho)\cdot e^{-\mu t}\cdot\rho\sum_{n=1}^{+\infty}\rho^{n-1}\frac{(\mu t)^{n-1}}{(n-1)!}$$

$n-1=k$:

$$= (1-\rho)\cdot\delta(t) + \mu\rho\cdot(1-\rho)\cdot e^{-\mu t}\cdot\sum_{k=0}^{+\infty}\frac{(\rho\cdot\mu t)^k}{k!}$$

$$\sum_{k=0}^{+\infty}\frac{x^k}{k!} = e^x$$

IT IS EXP. DISTRIBUTED!

$$= (1-\rho)\cdot\delta(t) + \rho\,\mu(1-\rho)\,e^{-\mu(1-\rho)t} \qquad \boxed{e^{-\mu t}\cdot e^{\mu\rho t}}$$

BECAUSE of EXP. DISTRIBUTION ⇒ in M/M/1 QUEUE

$$E\{n_W\} = \frac{\rho^2}{(1-\rho)}$$

$$E\{T_W\} = \underset{n=0}{\phi} + \rho\,\frac{1}{\mu(1-\rho)}$$

⇒ PDF of WAITING TIME IS EXP. DISTRIBUTED

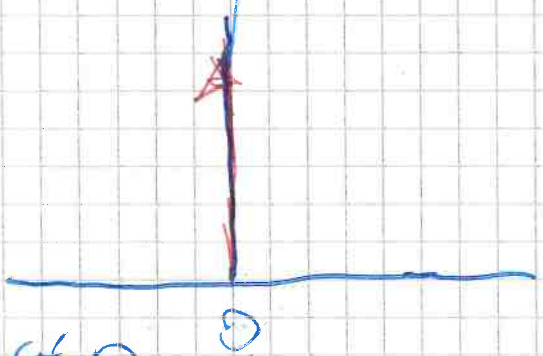
⇒ We can conclude that:

$$\boxed{E\{T_W\} = \frac{1}{\mu}\cdot\frac{\rho}{1-\rho}}$$

$\Longrightarrow$ LAST FEW STEPS EXPLAINED:

$$f_{T_W}(t) = (1-\rho)\cdot\delta(t) + \rho\,\underbrace{\mu(1-\rho)\,e^{-\mu(1-\rho)t}}_{\text{PDF of an EXPONENTIAL R.V.}}$$

$$\delta(t) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$



$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

PDF of an EXPONENTIAL R.V.

$$\lambda e^{-\lambda t}$$

$$E\{x\} = \frac{1}{\lambda}$$

$\Rightarrow$ In this case

$$E\{x\} = \frac{1}{\mu(1-\rho)}$$

$$\Rightarrow E\{T_W\} = \cancel{0} + \frac{\rho}{\mu(1-\rho)}$$

for $t \neq 0$

$$\Rightarrow E\{T_W\} = \frac{\rho}{\mu(1-\rho)}$$

And the PDF of the WAITING TIME is EXPONENTIALLY DISTRIBUTED.

$$g(t) \circledast \delta(t) = g(t)$$

PDF of the QUEUEING TIME IN M/M/1 QUEUES:

⇒ Now need to consider 2 R.V.s' PDF:

1) WAITING TIME'S PDF (Found before) 2) SERVICE TIME'S PDF

$$f_T(t) = \underbrace{\left[(1-\rho)\,\delta(t) + \mu(1-\rho)\cdot\rho\cdot e^{-\mu(1-\rho)t}\cdot u(t)\right]}_{\text{PDF of the WAITING TIME}} \circledast \underbrace{\left[\mu\cdot e^{-\mu t}\cdot u(t)\right]}_{\text{PDF of SERVICE TIME}}$$

$f_T(t) = f_{Tw}(t) \circledast f_{Ts}(t)$

$E\{T\} = E\{T_W\} + E\{T_S\}$

$$= \mu(1-\rho)\, e^{-\mu t}\cdot u(t) + \mu^2\cdot(1-\rho)\cdot\rho\left[e^{-\mu(1-\rho)t}\cdot u(t)\right] \circledast \left[e^{-\mu t}\cdot u(t)\right]$$

⇒ Again, apply ~~...~~ $\boxed{\mathcal{L}\text{-Transform}}$ to the CONVOLVED PART ⇒ we would get a MULTIPLICATION.

$$\cancel{= \mu(1-\rho)\left[\frac{1}{s+\mu} + \mu\rho\,\frac{1}{s+\mu(1-\rho)}\cdot\frac{1}{(s+\mu)}\right]}$$

$$f_T(s) = \frac{\mu(1-\rho)}{s+\mu} + \frac{\mu^2\cdot(1-\rho)\cdot\rho}{(s+\mu(1-\rho))}\cdot\frac{1}{s+\mu}$$

$$= \mu(1-\rho)\cdot\left[\frac{1}{s+\mu} + \frac{\mu\rho}{(s+\mu(1-\rho))}\cdot\frac{1}{s+\mu}\right]$$

$$= \mu(1-\rho) \cdot \frac{1}{s+\mu} \left[1 + \frac{\mu \rho}{s+\mu(1-\rho)} \right]$$

$$= \frac{\mu(1-\rho)}{\cancel{s+\mu}} \cdot \left[\frac{\cancel{(s+\mu)} - \cancel{\mu\rho} + \cancel{\mu\rho}}{s+\mu(1-\rho)} \right]$$

$$= \frac{\mu(1-\rho)}{s+\mu(1-\rho)}$$

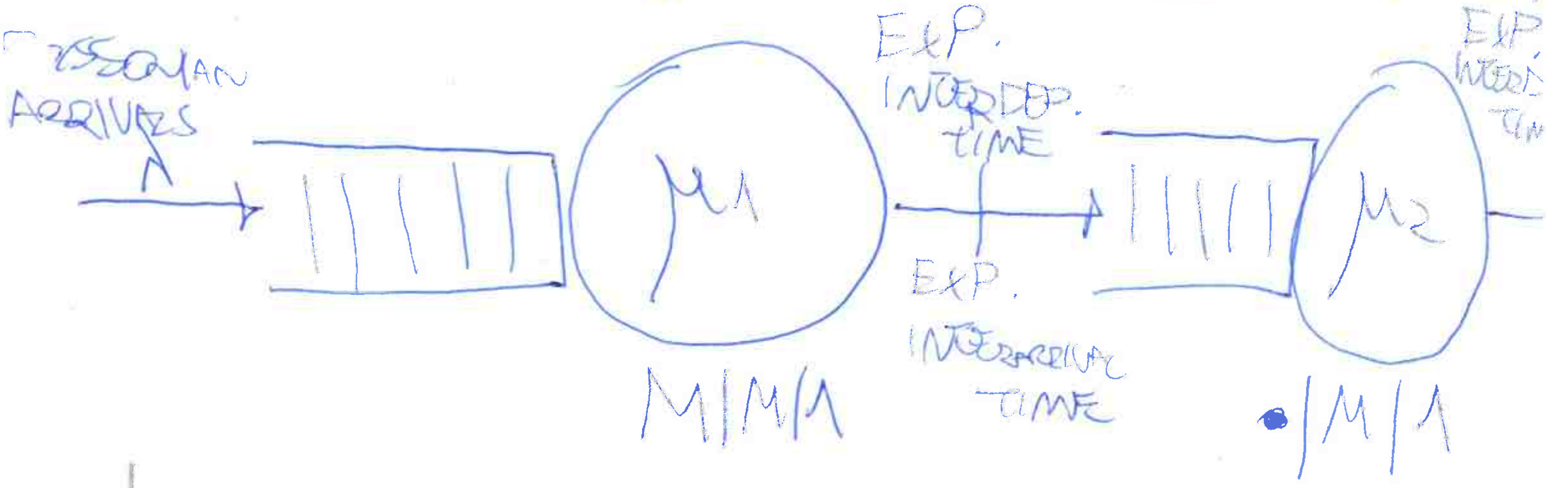
$$\mathcal{L}^{-1} \Rightarrow \boxed{f_T(t) = \mu(1-\rho) \cdot e^{-\mu(1-\rho) \cdot t} \cdot u(t)}$$

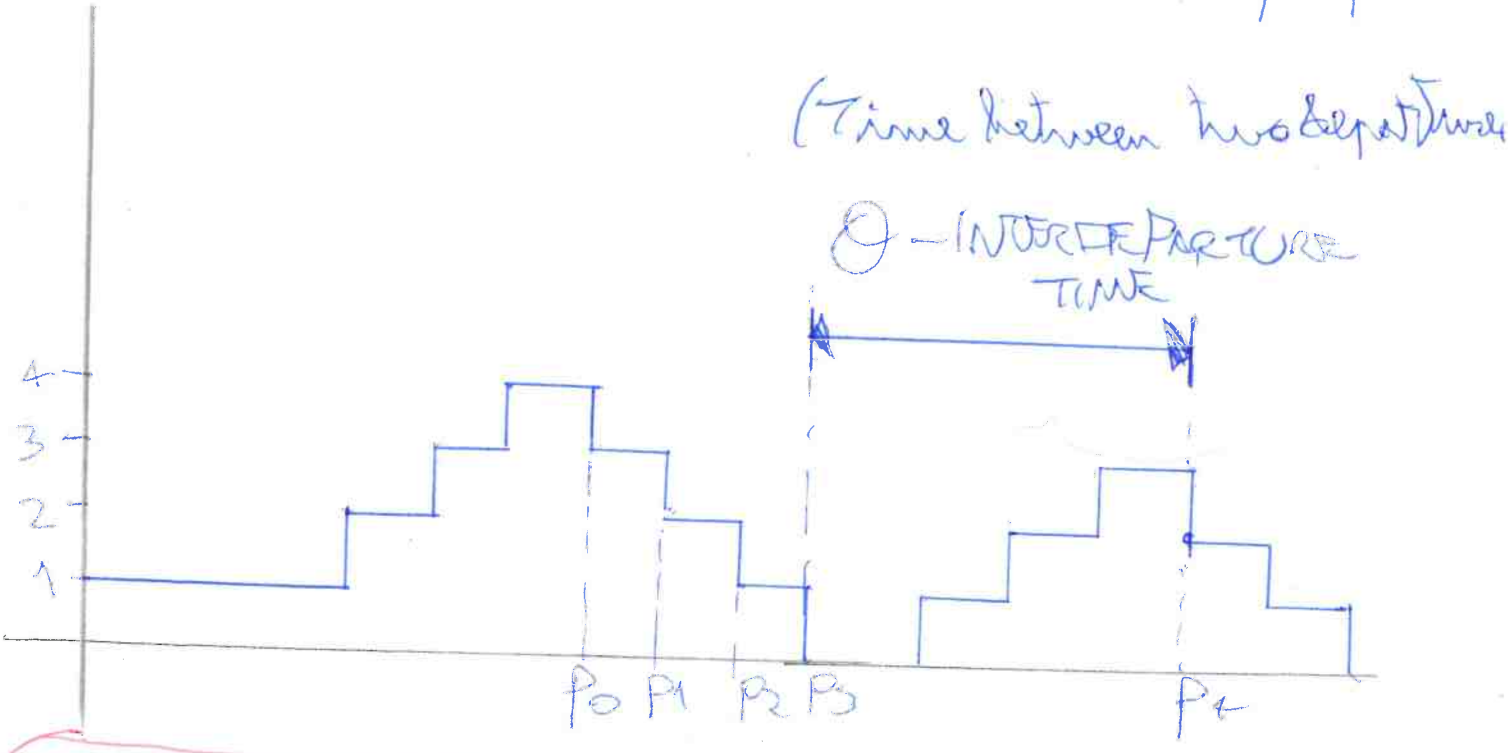
$\Rightarrow$ The PDF of the QUEUEING TIME in M/M/1 QUEUES is EXPONENTIALLY DISTRIBUTED.

$$\Rightarrow E\{n\} = \frac{\rho}{1-\rho}$$

$$E\{T\} = \frac{1}{\mu} \cdot \frac{1}{(1-\rho)} = \frac{\frac{1}{\mu}}{1-\rho}$$

61) NETWORKS of QUEUES - BURKE THEOREM





BURKE THEOREM'S THESIS:

"In M|M|NS queues at steady-state, the departure process is still POISSONIAN and is independent from the input one.

FULL PROOF of BURKE THEOREM (not what we do!)

We would need to show that, having POISSONIAN ARRIVALS, the interdeparture time is exponentially distributed and independent from the input one, in an M|M|NS queue.

OUR PROOF: In our case, we just show that, having POISSONIAN ARRIVALS, the departure process is still POISSONIAN in an M/M/1 QUEUE.

(≠ BURKE THEOREM!)

Θ = R.V. corresponding to the interdeparture time in M/M/1 queue at STEADY STATE.

Θ – 2 CASES

1° case – EMPTY QUEUE at departure: $P_0^{(d)}$

2° case – NON-EMPTY QUEUE: $1-P_0^{(d)}$

(*) because sum of the PDF of two independent R.V.s.

$$f(t\,|\,1st\ case) = \underbrace{\lambda e^{-\lambda t}\mu(t)}_{\text{new customer arriving}} \circledast \underbrace{\mu e^{-\mu t}\mu(t)}_{\text{new customer departing}}$$

(residual) ⇒ MEMORYLESS INTERARRIVAL

FULL RESIDUAL of SERVICE (EXP. distributed R.V.)

BUS STATION PARADOX (MEMORYLESS for exponential r.v.)

$$f(t\,|\,2nd\ case) = \mu e^{-\mu t}\mu(t)$$

[SERVICE-TIME]: finishing service = finishing depart.

Take the $\mathcal{L}$ transform of case ① and ②

$$F(s) = P_0^{(d)}\,\frac{\lambda}{s+\lambda}\,\frac{\mu}{s+\mu} + \left(1-P_0^{(d)}\right)\frac{\mu}{s+\mu}$$

$P_0^{(d)} = P_0^{(a)}$ (at arrival) since the queue has UNITARY VARIATIONS (as in M/G/1)

$P_0^{(a)} = P_0$ (of a random customer) thanks to PASTA PROPERTY. (POISSON ARRIVALS!)

$$p_0^{(d)} = p_0 = 1-\rho = 1-\frac{\lambda}{\mu}$$

$$1-p_0^{(d)} = \cancel{1}-\cancel{(1}-\rho) = \rho = \frac{\lambda}{\mu}$$

$$\Rightarrow F(s) = \left[1-\frac{\lambda}{\mu}\right]\frac{\lambda\cdot\mu}{(s+\lambda)(s+\mu)} + \frac{\lambda}{\cancel{\mu}}\frac{\cancel{\mu}}{(s+\mu)}$$

$$= \frac{\mu-\lambda}{\cancel{\mu}}\frac{\lambda\cdot\cancel{\mu}}{(s+\lambda)(s+\mu)} + \frac{\lambda}{s+\mu}$$

$$= \frac{\lambda}{s+\mu}\cdot\left[\frac{\mu-\lambda}{(s+\lambda)}+1\right]$$

$$= \frac{\lambda}{s+\mu}\cdot\left[\frac{\mu-\cancel{\lambda}+s+\cancel{\lambda}}{s+\lambda}\right]$$

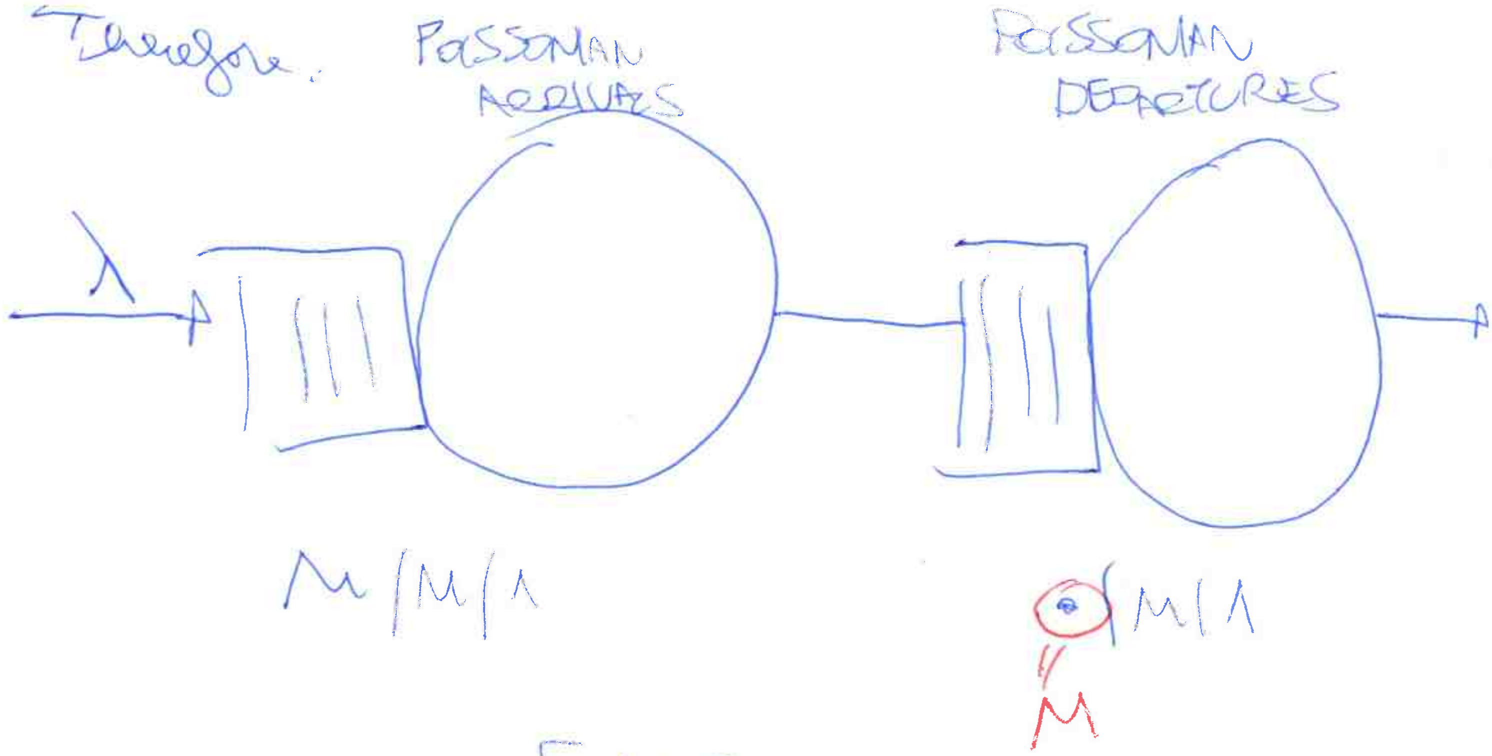
$$= \frac{\lambda}{\cancel{(s+\mu)}}\cdot\frac{\cancel{(s+\mu)}}{(s+\lambda)}$$

$$= \frac{\lambda}{s+\lambda}$$

$$\Rightarrow g(t) = \lambda\cdot e^{-t\lambda}\mu(t)$$

$\Rightarrow$ INTER DEPARTURE TIME is indeed EXP. DISTRIBUTED!

Therefore:



Also, remembering: [NO FEEDBACK!]

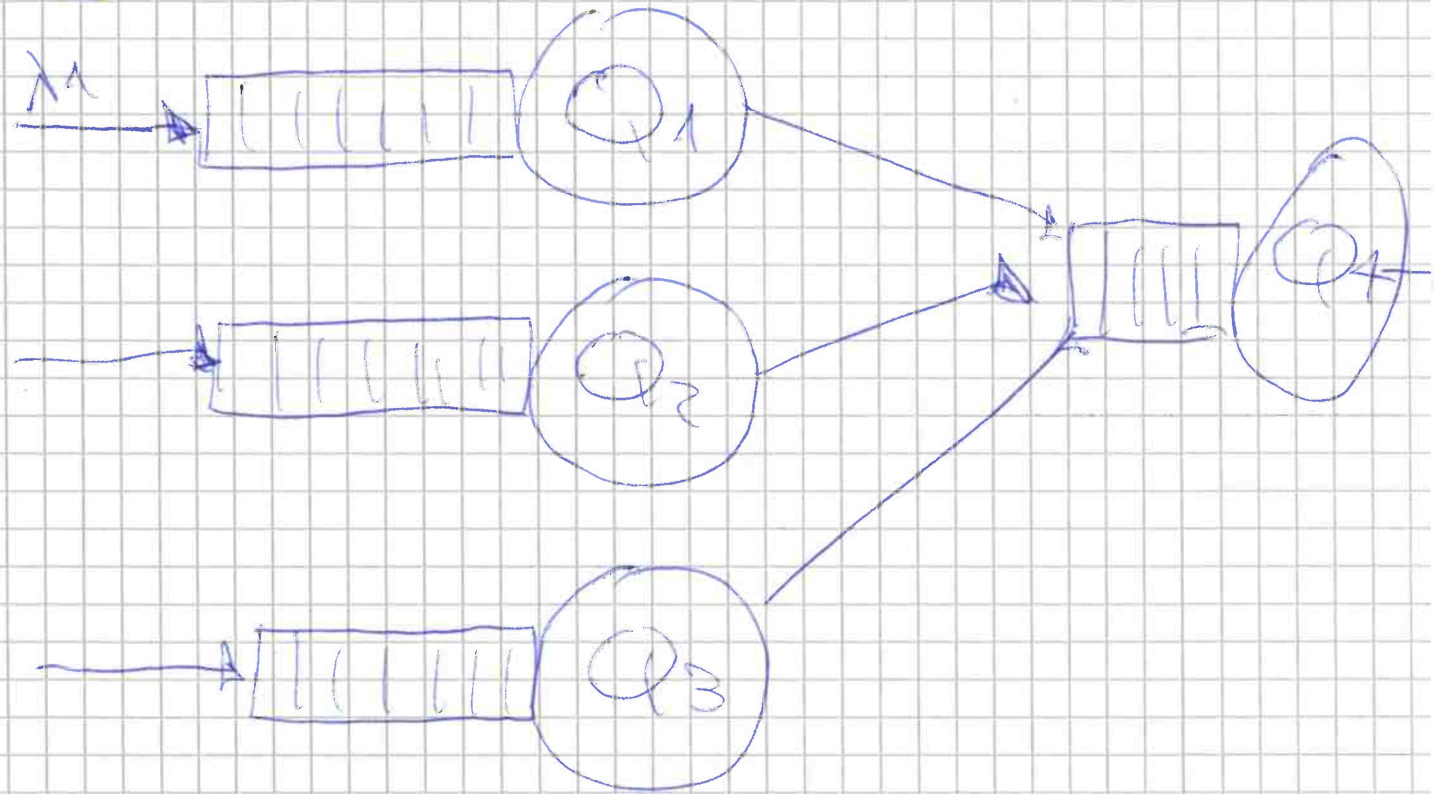
- STATISTICAL DECOMPOSITION of a POISSON PROCESS into (n) POISSON PROCESSES
- COMPOSITION of (n) POISSON PROCESSES is still a POISSON PROCESS
- BURKE THEOREM

⇓

We can only study OPEN MARKOVIAN NETWORKS of queues without feedback.

If we have the result of JACKSON THEOREM we can study OPEN MARKOVIAN NETWORKS of queues with feedback. (i.e.: treat non-Markovian queues as if they were Markovian!)

6.2) DEFINITION of OPEN MARKOVIAN NETWORK of QUEUES.



OPEN NETWORK of QUEUES: Network where we have customers coming in from the "EXTERNAL WORLD" and going out to the external world (i.e. entering & leaving the queue).

a) STATE: $\underline{N} = (n_1, n_2, n_3, n_4)$

→ # USERS in every single QUEUE of the network

Q = ~~# CUSTOMERS in each queue~~ # QUEUES in the network.

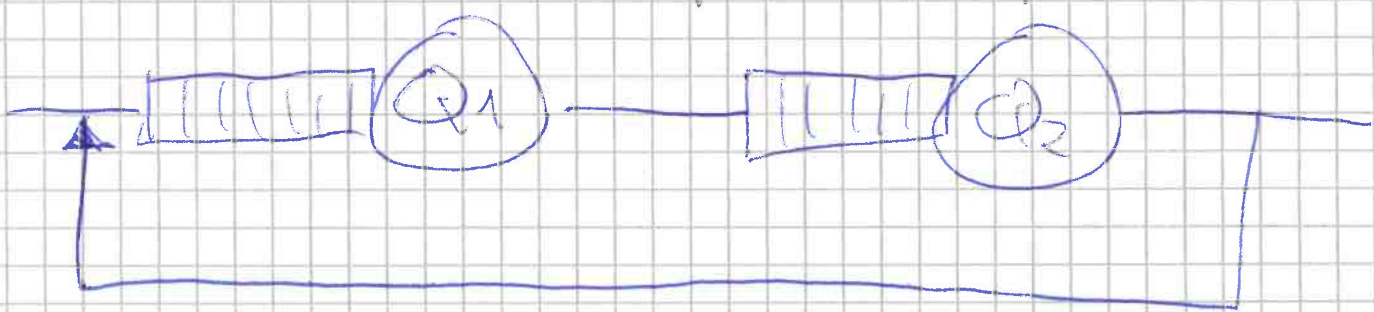
NB: The STATE EVOLUTION process of a Markovian network of QUEUES is a Markov Chain (B&D in Q dimensions)

b) STATE PROBABILITY of a NETWORK:

It is given by the PRODUCT-FORM SOLUTION.

$$\boxed{P_{k_1,k_2,\dots,k_n} = P_{k_1} \cdot P_{k_2} \cdots P_{k_n}}$$ STATE PROBABILITY

EXAMPLE:



$$P_{k_1 k_2} = \left(1 - \frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_1}\right)^{k_1} \cdot \left(1 - \frac{\lambda}{\mu_2}\right) \cdot \left(\frac{\lambda}{\mu_2}\right)^{k_2}$$

(c) ERGODICITY CONDITION of an Open Markovian Network of QUEUES

$$\boxed{\lambda < \min(\mu_1, \mu_2)}$$ [FOR 2-QUEUE network]

Ex: $\mu_1 = 2\mu$, $\mu_2 = 5\mu$

$\Rightarrow \boxed{\lambda < 2\mu}$ $\frac{\lambda}{2\mu} < 1$ $\frac{\lambda}{5\mu} < 1$ THIS IS THE CONDITION

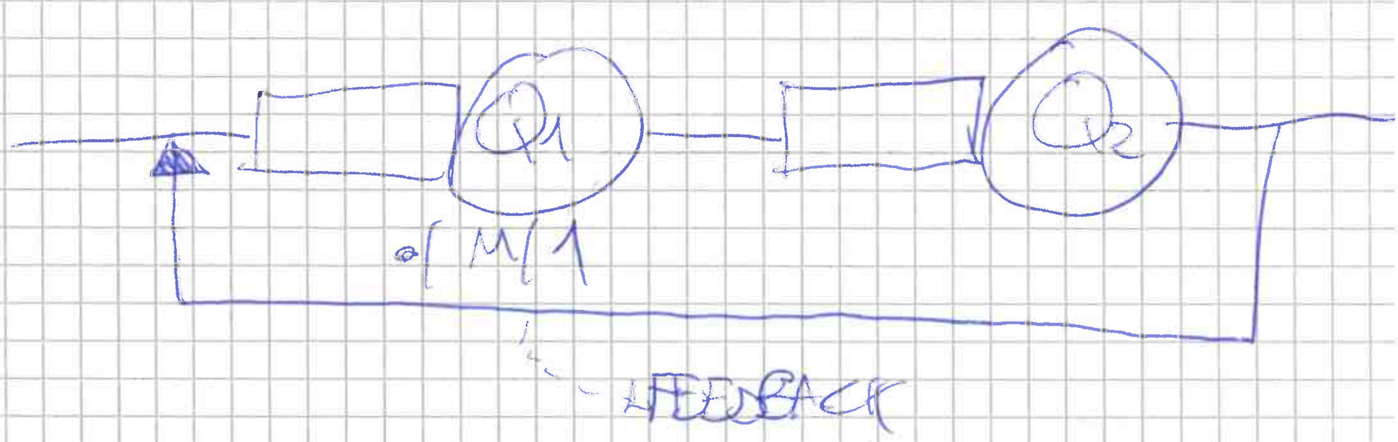
OPEN MARKOVIAN NETWORK of QUEUES VS OPEN NETWORK of QUEUES

(63) An Open Markovian Network of Queues is an open network of MARKOVIAN Queues only if every QUEUE is MARKOVIAN (i.e. This only occurs if all the Markovian queues are without FEEDBACK).

(d) FEEDBACK VS NO FEEDBACK

A network of QUEUES with feedback has queues with "ARR... " ... ...

⇒ EXAMPLE: TANDEM



However, this prevents the Markovian nature from existing ⇒ NON-MARKOVIAN QUEUE!

⇒ the STATE no longer depends only on the current #CUSTOMERS, but also on the current "PHASE".

⇒ No independent arrivals!

We still have an OPEN MARKOVIAN NETWORK of QUEUES / with FEEDBACK [EVEN if the single queues are NOT MARKOVIAN and INDEPENDENT]

In this case,
⇒ We do have a: [THANKS TO JACKSON?]
OPEN MARKOVIAN NETWORK of QUEUES,
but not an:
OPEN NETWORK of MARKOVIAN QUEUES!

The JACKSON

THEOREM tells us that

we can study ~~[illegible]~~ NETWORKS of

non-MARKOVIAN QUEUES (without feedback)

as if their queues were MARKOVIAN!

04 RESULT OF JACKSON THEOREM:

↳ TRAFFIC EQUATIONS!

They are valid in STEADY-STATE and are used to solve the PRODUCT FORM SOLUTION of form (for MARKOVIAN / NON-MARKOVIAN QUEUES)

$$\boxed{P_{k_1 k_2} = \left(1-\frac{\lambda_1}{\mu_1}\right)\cdot\left(1-\frac{\lambda_2}{\mu_2}\right)\cdot\left(\frac{\lambda_1}{\mu_1}\right)^{k_1}\cdot\left(\frac{\lambda_2}{\mu_2}\right)^{k_2}}$$

TRAFFIC | FREQUENCY EQUATIONS.

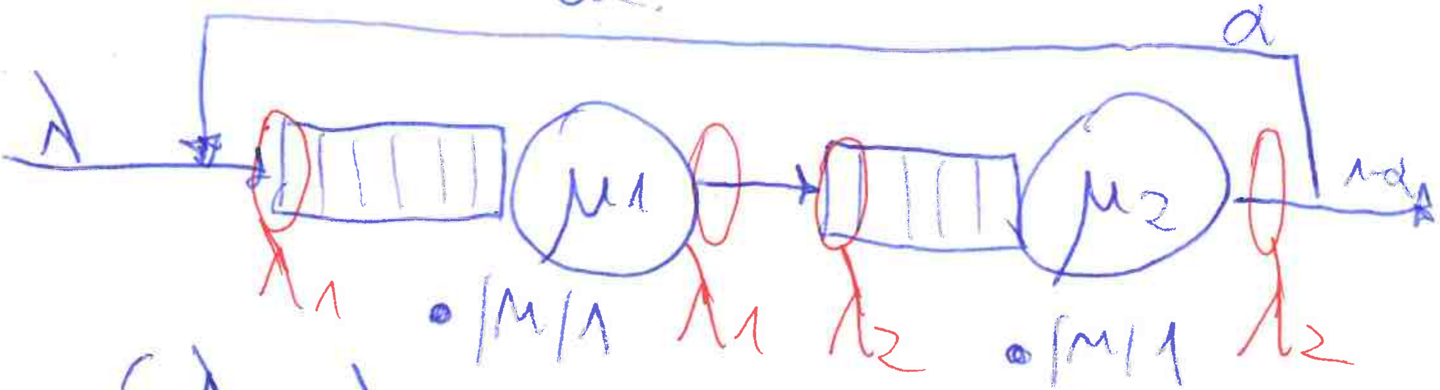
INPUT frequency of arrivals [AT STEADY STATE]

=

OUTPUT frequency of departure.

⇒ They allow us to find λ_1, λ_2 (as a function of Λ / λ).

EXAMPLE USAGE:



$$\begin{cases} \lambda_1 = \lambda_2 \\ \lambda_1 = \lambda + \alpha\lambda_2 \end{cases}$$

[In exercises, we're using Λ_1, Λ_2 though]

$\Rightarrow \lambda_1 = \lambda + \alpha\lambda_1$

$\Rightarrow \lambda_1(1-\alpha) = \lambda \Rightarrow \boxed{\lambda_1 = \frac{\lambda}{(1-\alpha)}} \boxed{= \lambda_2}$ $\lambda_2 < \mu_2$, $\lambda_1 < \mu_1$

$\Rightarrow$ We can then use the found λ_1, λ_2 to solve the ~~TRAFFIC EQUATIONS~~ PRODUCT-FORM SOLUTION

INTERESTING RESULT of JACKSON'S THEOREM:

[As ~~we are~~ if they were considering the queues as MARKOVIAN & independent]

$\Rightarrow$ The PRODUCT-FORM holds for non-MARKOVIAN QUEUES too (the ones with feedback)

(64) REQUIREMENTS of the JACKSON THEOREM

It can be applied to OPEN MARKOVIAN NETWORKS OF QUEUES

OPEN MARKOVIAN NETWORKS OF QUEUES (NO FEEDBACK)
⇓
Queues are Markovian

OPEN MARKOVIAN NETWORKS OF QUEUES (WITH FEEDBACK)
⇓
As if queues were Markovian

Same PRODUCT-FORM STRUCTURE & TRAFFIC EQUATIONS

REQUIREMENTS for a JACKSON NETWORK OF QUEUES

If an Open Markovian Network of Queues respects the following properties, then it is said to be a **JACKSON NETWORK of QUEUES**.

- Single class of users
- Infinite WAITING LINE, $N_W = \infty$ (Ex: $M/M/1/\infty$)
- Generic # QUEUES in the Network $= Q$
- Multiple servers $N_{S\,i}$ in each queue Q_i.
  $\Rightarrow M/M/m_i$ for Q_i.
- EXPONENTIAL SERVICE TIME in each queue Q_i.
  $$\boxed{E\{T_{S\,i}\} = \frac{1}{\mu_i}} \text{ in } Q_i.$$
- POISSON ARRIVALS from external world:
  γ_i
- FCFS SCHEDULING / SERVING POLICY
- $r_{ij} \Rightarrow$ Routing probability from Q_i to Q_j.

⇒ JACKSON NETWORK of QUEUES if all the requirements are satisfied.

PROBABILITY to "GO OUT":

$$r_{i0} = 1 - \sum_{k=1}^{Q} r_{i,k}$$

everything entering Queue (i) from other queues (j)

TRAFFIC EQUATIONS' FORM:

$$\boxed{\lambda_i = \gamma_i + \sum_{j=1}^{Q} \lambda_j \cdot r_{ji}} \quad \forall i, i \in [1, Q]$$

everything entering queue (i) from the external world

ERGODICITY (STEADY-STATE) CONDITION:

The network will be in STEADY-STATE only if every queue is ERGODIC.

$$\boxed{\lambda_i < m_i \cdot \mu_i} \quad \forall i, i \in [1, Q]$$

$$\Rightarrow \boxed{\frac{\lambda_i}{\mu_i} < m_i} \qquad \rho_i = \frac{E\{n_{s_i}\}}{N_{s_i}} = \frac{\lambda_i}{\mu_i \cdot m} < 1$$

$$A < N_s$$

$$\Rightarrow \lambda < \min(\mu_1, \mu_2, \dots, \mu_n)$$

(a) How to OPERATE with the JACKSON THEOREM:

STEP (1):

a) Lay out the TRAFFIC EQUATIONS,
b) Solve the TRAFFIC EQUATIONS
c) Verify that every queue is ERGODIC.

STEP (2):

Consider each queue as an $M|M|m_i$ QUEUE [TRUE if NO feedback, not TRUE if NO feedback].

For an $M/M/N_S$ QUEUE:

$$P_n = \begin{cases} \left(\frac{\Lambda_i}{\mu_i}\right)^n \cdot \frac{\Lambda}{n!} \cdot P_0 & n \leq N_S \\ \left(\frac{\Lambda_i}{\mu_i}\right)^n \cdot \frac{\Lambda}{N_S!} \cdot \frac{\Lambda}{N_{S_i}^{n-N_S}} P_0 & n > N_S \end{cases}$$

SIMPLE DEFINITION of ρ_i

$$\rho_i = \frac{\lambda_i}{m_i \mu_i} \Rightarrow \frac{\lambda_i}{\mu_i} = \rho_i \cdot m_i$$

$$P_i(k_i) = \begin{cases} p_i(0) \cdot \frac{(\rho_i \cdot m_i)^{k_i}}{k_i!} & k_i \leq m_i \\ p_i(0) \cdot \frac{(\rho_i \cdot \cancel{m_i})^{k_i}}{m!} \, m_i^{m_i} = N_S^{N_S} & k_i > m_i \end{cases}$$

$$\rho_i = \left(\frac{\Lambda}{\mu \cdot N_S}\right)^n$$

$$\cancel{\frac{(\rho_i \, m_i)^{k_i}}{m_i^{k_i}}}$$

STEP (3): with no P_0 4

Multiply the STATE PROBABILITY p_n with one another to obtain $P_{\underline{N}}$ (PRODUCT-FORM SOLUTION)

$$\boxed{P_{\underline{N}} = \prod_{i=1}^{N} p_i} \quad \forall i \in [1, Q]$$

b) FEELING OF PROOF OF JACKSON THEOREM

We want to obtain:

$$\underline{P}(n) = P_{k_1}, P_{k_2}, \ldots\ldots, P_{k_N}$$

Instead of building a COMPLEX multi-dimensional STATE TRANSITION DIAGRAM, we only consider a CONFIGURATION of states and then take the possible TRANSITIONS between them.

↳ We hence consider the following cases:

1. Someone enters the NETWORK of QUEUES from the EXTERNAL WORLD
2. Someone leaves the NETWORK of QUEUES and goes to the EXTERNAL WORLD
3. Someone moves within the NETWORK of QUEUES, i.e: internal TRANSITION

JACKSON'S THEOREM DEMONSTRATION

[of the PRODUCT-FORM SOLUTION

~~$P_N = P_{k1} \cdot P_{k2} \cdot \ldots \cdot P_k$~~

result of JACKSON'S THEOREM

1st-queue, i-th-queue, j-th-queue

$$\boxed{P_N = P_{k_1, \ldots, k_i, \ldots, k_j, \ldots, k_N}}$$

VECTOR containing the #CUSTOMERS in each one of the N queues.

DEF: Jackson, instead of trying to build a STATE TRANSITION DIAGRAM [MULTI-DIMENSIONAL] for a general open Markovian Network of queues, he considered only some GENERIC CONFIGURATIONS of STATES

[~~...~~ ⇒ Only certain TRANSITIONS (3) are indeed possible!]

3 POSSIBLE TRANSITIONS are possible within a NETWORK ⇒ Take 4 CONFIGURATIONS ~~...~~ of states corresponding to these situations:

1. **INITIAL SITUATION**:

$$\underline{N} = (k_1, \ldots, k_i, \ldots, k_j, \ldots, k_N)$$

2. **SOMEONE ARRIVING to QUEUE** i FROM EXTERNAL

$$\underline{N}_{i,0} = (k_1, \ldots, k_i + 1, \ldots, k_j, \ldots, k_N)$$

(entrance from external to i)

3. SOMEONE LEAVING FROM QUEUE j TO EXTERNAL

$\underline{N}_{0,j} = (k_1, \dots, k_i, \dots, k_{j-1}, \dots, k_N)$

4. SOMEONE (INTERNALLY) TRANSITIONING FROM QUEUE j TO QUEUE i

$\underline{N}_{i,j} = (k_1, \dots, k_{i+1}, \dots, k_{j-1}, \dots, k_N)$ $(i \neq j)$

$k_j - 1 \geq 0$

$\Rightarrow$ TRANSITION TYPES

(1) ENTERING TRANSITION

(1) $\underline{N}_{0,j} \to \underline{N}$ SOMEONE (A CUSTOMER) ~~LEAVING FROM~~ ENTERS from EXTERNAL TO QUEUE j.

$k_{j-1} \to k_j$

$\Downarrow$

POISSON PROCESS (γ_j) entering intensity.

(1) $\underline{N} \to \underline{N}_{i,0}$ SOMEONE ENTERS from EXTERNAL to QUEUE i

$k_i \to k_{i+1}$

$\Downarrow$

POISSON PROCESS (γ_i) entering intensity

(2) EXITING TRANSITION

(2) $\underline{N}_{i,0} \to \underline{N}$ (A CUSTOMER) SOMEONE EXITS from the NETWORK (FROM QUEUE i to the EXTERNAL)

$k_{i+1} \to k$

(2) $\underline{N} \to \underline{N}_{0,j}$ FROM QUEUE j to the EXTERNAL

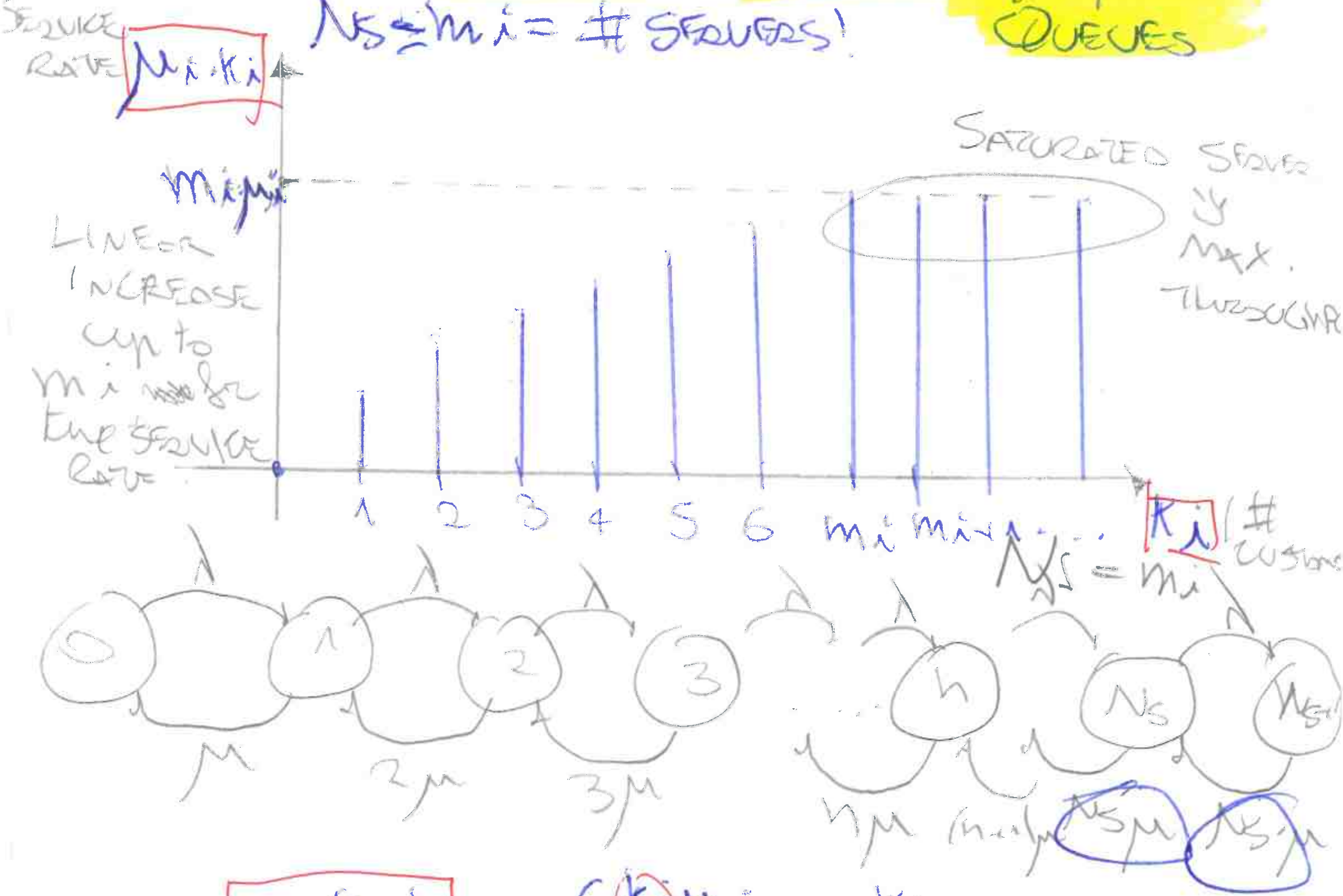
$k_j \to k_j - 1$

EXITING from QUEUE i / QUEUE is due to CUSTOMER SERVICING $\Rightarrow$ (2) and 2

$\Downarrow$

Consider SERVICE RATE in $M/M/m_i$ QUEUES

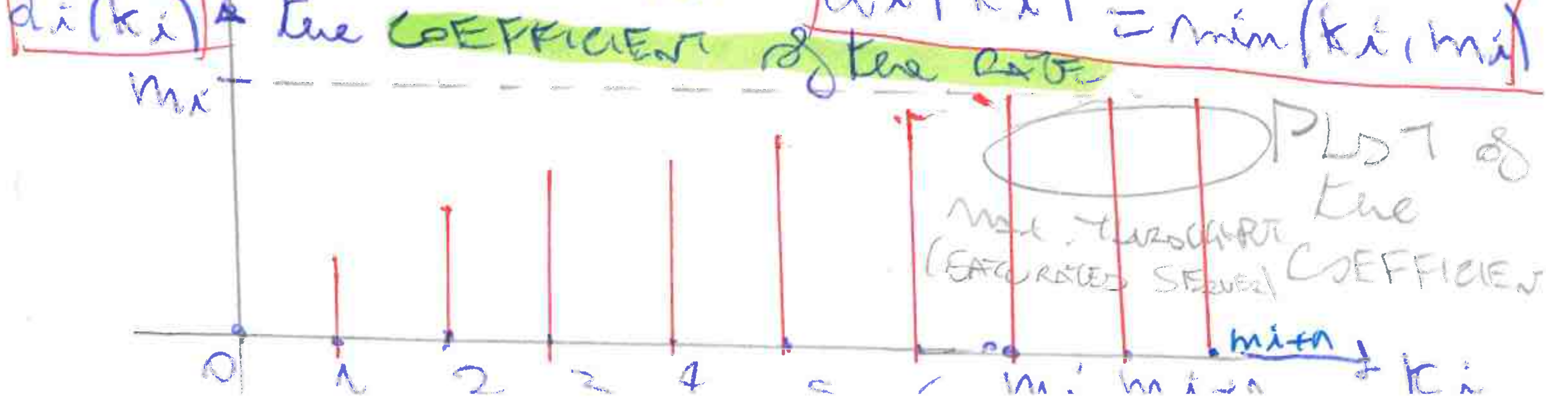
$N_S = m_i$ = # SERVERS!



$$\boxed{\mu_i(k_i)} = \begin{cases} k_i \mu_i & k_i \le m_i \quad \text{LINEAR INCREASE} \\ m_i \mu_i & k_i > m_i \quad \text{SATURATED SERVER} \end{cases}$$

$m_i = N_S$

$\Rightarrow$ Consider hence: $\boxed{d_i(k_i) = \min(k_i, m_i)}$ the COEFFICIENT of the RATE



NB: In case of TRANSITION (2) (before making the transition) there are $\boxed{k_i + 1 \text{ CUSTOMERS}}$ in QUEUE Q_i.

(2) → EXITING PROBABILITY (from the network) $= \boxed{\alpha_i(k_i+1)\,\mu_i\,dt\,r_{i0}}$

(2) $\alpha_j(k_j)\,\mu_j\,dt\cdot r_{j0}$ where $\boxed{r_{i0} = 1 - \sum_{j=1}^{Q} r_{ij}}$

(3) INTERNAL MOVEMENT (Same # CUSTOMERS in the system)

(3) $\underline{N}_{i,j} \rightarrow \underline{N}$

$k_i+1, k_j-1 \rightarrow k_i, k_j$

SOMEONE (1 CUSTOMER) MOVES FROM QUEUE i TO QUEUE j.

→ Probability of INTERNAL MOVEMENT $= \boxed{\alpha_i(k_i+1)\,\mu_i\,dt\,r_{ij}}$

(5) $\underline{N} \rightarrow \underline{N}_{i,j}$ CUSTOMER MOVING FROM QUEUE j TO QUEUE i

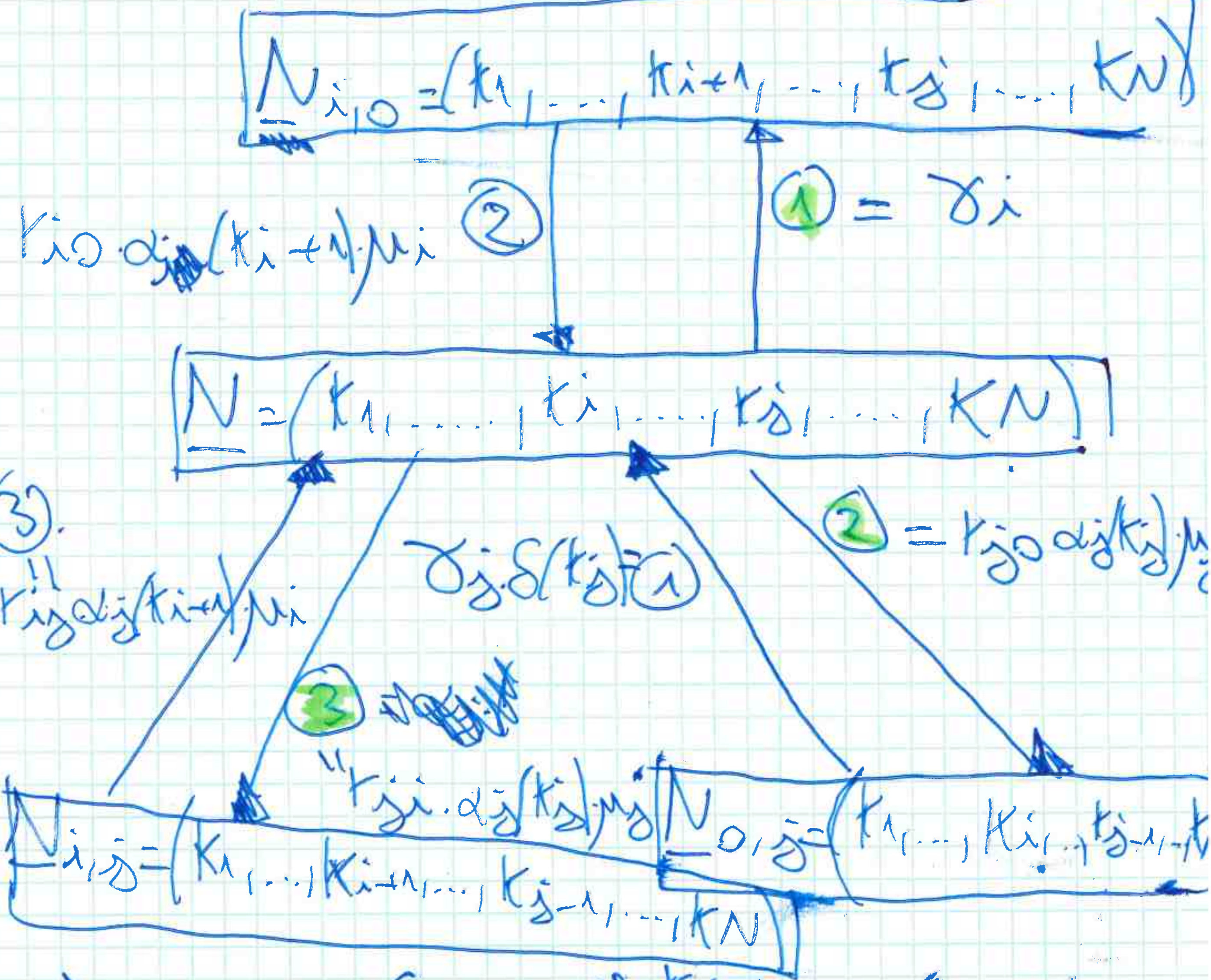
$k_i, k_j \rightarrow k_i+1, k_j-1$

Where $\delta(k_i) = \begin{cases} 1 & \text{if } k_i \geq 1 \text{ (nobody leaving)} \\ 0 & \text{OTHERWISE} \end{cases}$

DELTA of KRONECKER

(can never have less than 1 customer!)

⇒ We can then visualize such transitions:



$$\Rightarrow \underbrace{\delta(k_j)}_{\text{DELTA of KRONECKER}} = \begin{cases} 1 & \text{if } k_j \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

⇒ JACKSON'S SMART IDEA: We can now apply the FCP to $P_{\underline{N}}$ for the EXIT FLOW

$$P_{\underline{N}} \left[\sum_{i=1}^{N} \gamma_i + \sum_{j=1}^{N} \left[\sum_{i=1}^{N} r_{ji}\,\alpha_j(k_j)\mu_j + \alpha_j(k_j)\mu_j\, r_{j0} \right] \right.$$

① from EXTERNAL ③ INTERNAL ② TO EXTERNAL

ARRIVALS to i FROM OTHER STATES j to i & ...

Because of POISSONIAN INPUTS from the EXTERNAL WORLD, we would have INFINITELY many states!

→ bezieht sich nur auf den ersten Term!

$$P_{\underline{N}}\left[\sum_{i=1}^{N}\gamma_i+\sum_{j=1}^{N}\left[\sum_{i=1}^{N}\alpha_j(k_j)\mu_j\cdot r_{ji}+\alpha_j(k_j)\mu_j\cdot r_{j0}\right]\right]$$

$$P_{\underline{N}}\left[\sum_{i=1}^{N}\gamma_i+\sum_{j=1}^{N}\alpha_j(k_j)\mu_j\left[\sum_{i=1}^{N}r_{ji}+r_{j0}\right]\right]$$

$$\Rightarrow \underbrace{P_{\underline{N}}\left[\sum_{i=1}^{N}\gamma_i+\sum_{j=1}^{N}\alpha_j(k_j)\mu_j\right]}_{\text{EXIT FLOW}} \qquad \sum_{i=0}^{N}r_{ji}=1$$

EXIT FLOW FROM $P_{\underline{N}}$

⇒ Now consider the $\boxed{\text{INPUT FLOW}}$ ①/②/③ INTO $P_{\underline{N}}$

$$\sum_{i=1}^{N}P_{\underline{N}_{i,0}}\,\alpha_i(k_i+1)\mu_i\cdot r_{i0}+\sum_{j=1}^{N}P_{\underline{N}_{0,j}}\,\gamma_j\cdot\delta(k_j)$$

② ①

$$+\sum_{i=1}^{N}\sum_{j=1}^{N}P_{\underline{N}_{i,j}}\cdot\alpha_i(k_i+1)\mu_i\cdot r_{ij}$$

③

SOLUTION:

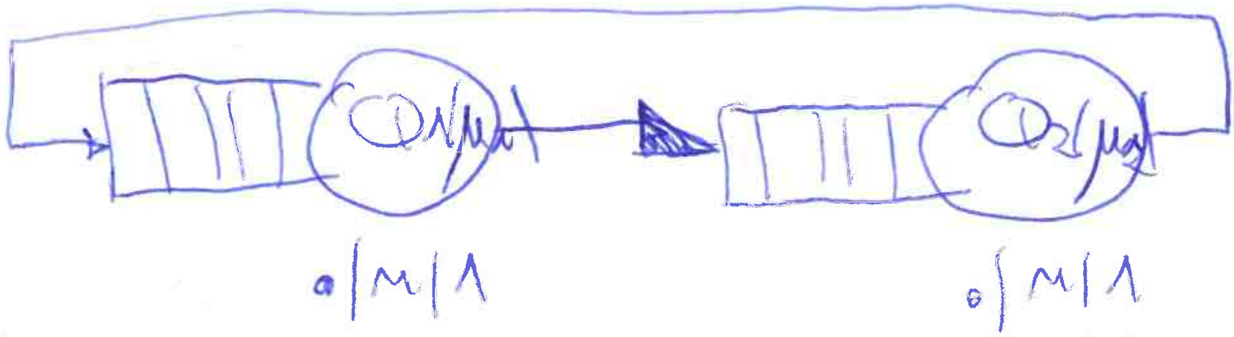
$\Gamma_{\underline{N}}$ EXIT FLOW = $\Gamma_{\underline{N}}$ INPUT FLOW

⇒ We can then obtain the PRODUCT-FORM SOLUTION from these GLOBAL BALANCE EQUATIONS for an OPEN MARKOVIAN NETWORK of QUEUES with NO FEEDBACK (20 pages of COMPUTATIONS!)

CLOSED MARKOVIAN NETWORK OF QUEUES:

OPEN: We do have INPUT ARRIVALS / customers leaving the system

CLOSED: having CONSTANT (same) # CUSTOMERS in the SYSTEM! [no INPUT ARRIVALS / OUTPUT exiting CUSTOMERS]



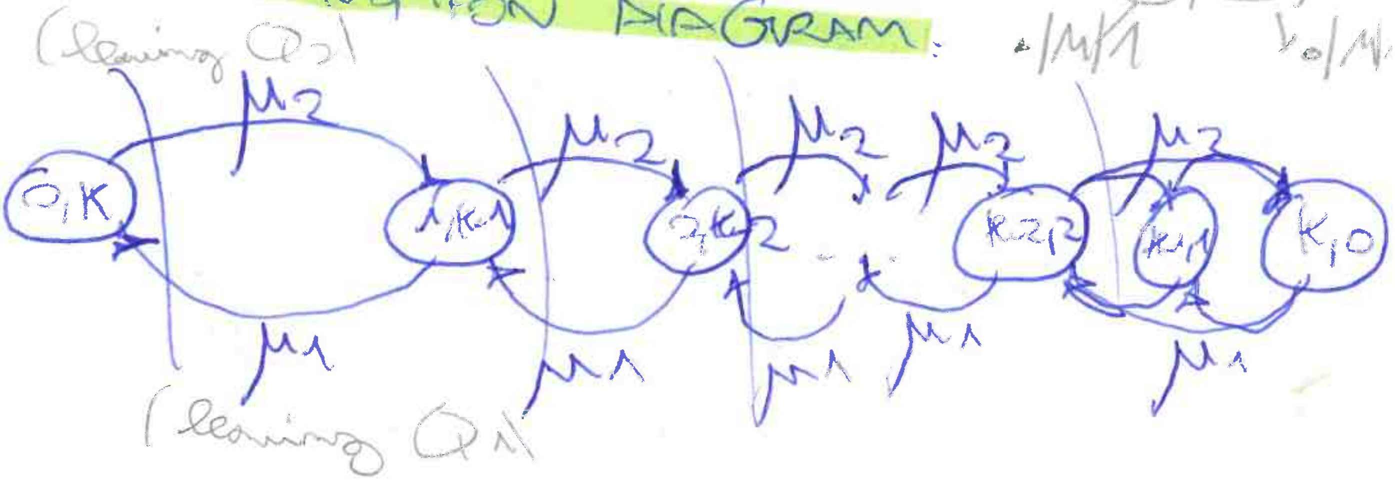
$\boxed{\text{STATE}} = \boxed{K}$ ⇒ MARKOVIAN network of queues.

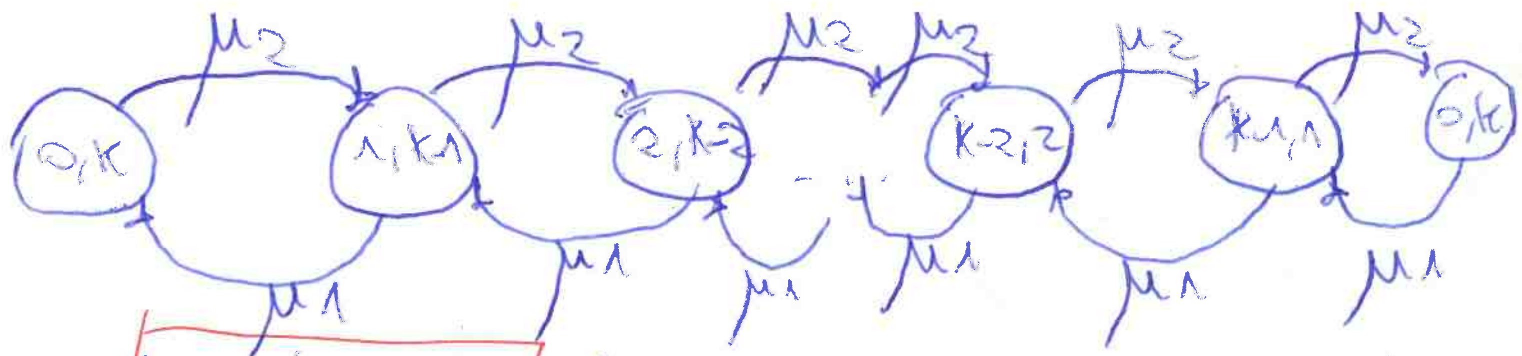
⇒ So, the state is ONLY characterized by the $\boxed{\text{\# CUSTOMERS in the system}}$

CUSTOMERS in Q_1 (n_1, n_2) # CUSTOMERS in Q_2

⇒ Also need EXPONENTIAL DISTRIBUTION of the SERVICE CENTER to have a MARKOVIAN network of queues.

STATE TRANSITION DIAGRAM: $\underline{n} = (n_1, n_2)$ $\cdot/M/1$ $\cdot/M/1$





$\boxed{n_1 + n_2 = K}$ (constant # CUSTOMERS in the queue)

⇒ Apply FCP to find the STATE PROBABILITY

Γ_0: $\mu_2 \cdot p_{0,K} = \mu_1 \cdot p_{1,K-1} \Rightarrow p_{1,K-1} = \frac{\mu_2}{\mu_1} \cdot p_{0,K}$

Γ_1: $\mu_2 \cdot p_{1,K-1} = \mu_1 \cdot p_{2,K-2} \Rightarrow p_{2,K-2} = \frac{\mu_2}{\mu_1} \cdot p_{1,K-1} = \left(\frac{\mu_2}{\mu_1}\right)^2 p_{0,K}$

Γ_2: $\mu_2 \, p_{2,K-2} = \mu_1 \cdot p_{3,K-3} \Rightarrow p_{3,K-3} = \frac{\mu_2}{\mu_1} p_{2,K-2} = \left(\frac{\mu_2}{\mu_1}\right)^3 p_{0,K}$

$$\Rightarrow \boxed{p_{i,K-i} = \left(\frac{\mu_2}{\mu_1}\right)^i \cdot p_{0,K}} \qquad 0 \leq i \leq K$$

⇒ Apply the NORMALIZATION CONDITION:

$$\boxed{\sum_{i=0}^{K} p_{i,K-i} = 1}$$

$$\Rightarrow \sum_{i=0}^{K} \left(\frac{\mu_2}{\mu_1}\right)^i \cdot p_{0K} = 1$$

$$\Rightarrow p_{0K} = \frac{1}{\sum_{i=0}^{K} \left(\frac{\mu_2}{\mu_1}\right)^i} = \frac{1 - \frac{\mu_2}{\mu_1}}{1 - \left(\frac{\mu_2}{\mu_1}\right)^{K+1}}$$

⇒ STATE PROBABILITY is then:

$$\boxed{P_{i,K-1} = \frac{1 - \frac{\mu_2}{\mu_1}}{1 - \left(\frac{\mu_2}{\mu_1}\right)^{K+1}} \cdot \left(\frac{\mu_2}{\mu_1}\right)^i} \quad 0 \le i \le K$$

[With FINITE # CUSTOMERS in the WAITING LINE].

* Same STATE PROBABILITY as an $M/M/1/N_W$ ⇒ STRONG SIMILARITY TO IT!

⇒ Just set $N_W = K-1$

$\boxed{\lambda = \mu_2}$ $\boxed{\mu = \mu_1}$ ⇒ $K = N_W + 1$

⇒ ~~$P_{i,K-1} =$~~

$$P_{i,N_W} = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{N_W+2}} \left(\frac{\lambda}{\mu}\right)^i$$

⇒ How can we actually solve the PRODUCT-FORM SOLUTION & find the STATE PROBABILITY?

⇓

GORDON-NEWELL!
THEOREM)

(66) GORDON - NEWELL THEOREM

For CLOSED Markovian Networks of QUEUES in STEADY-STATE.

→ PRODUCT-FORM!

[JACKSON ⇒ Traffic EQUATIONS]

EXAMPLE - M/M/1,

$$P_i(n) = (1-\rho_i)\cdot\rho_i^{n_i}$$

⇒ GORDON-NEWELL THEOREM $\neq$ JACKSON THEOREM.

In this case, for GORDON-NEWELL

$$\boxed{P_{k_1 k_2} = \frac{1}{G}\cdot\left(\frac{\lambda_1}{\mu_1}\right)^{k_1}\cdot\left(\frac{\lambda_2}{\mu_2}\right)^{k_2}}$$

P_0 ⇒ No longer NORMALIZATION CONDITION!

JACKSON THEOREM ($\boxed{\text{Open}}$ Markovian Networks of QUEUES)

↳ # QUEUES = $[0, \infty)$

↳ STATE PER QUEUE (# CUSTOMERS per QUEUE) $[0, \infty]$

Because they coming in from EXTERNAL

⇒ $\boxed{P_0}$ is a NORMALIZATION COEFFICIENT.

⇒ P_0^i (OPEN) in the PRODUCT-FORM SOLUTION

GORDON-NEWELL
CLOSED MARKOVIAN NETWORK OF QUEUES.

2 ▶ $\boxed{\text{STATE per QUEUE}}$ (when having K customers in the network of queues).

⇒ STATE = $[0, K_1]$

⇒ P_0^i only if going from 0 to ∞ as NORMALIZATION COEFFICIENT (JACKSON ⇒ Multiplication)

⇒ Need a different NORMALIZATION.

⇒ Yet, same shape of DECAY (GEOMETRIC)

(JACKSON ⇒ "Simple" multiplication)
G-N ⇒ Need a different NORMALIZATION COEFFICIENT based on the # CUSTOMERS P_0^i NORMALIZATION COEF.)

(2)

GORDON-NEWELL THEOREM
- DEMONSTRATION:

We are interested in the STATE EVOLUTION of the GORDON-NEWELL MARKOVIAN CLOSED NETWORKS OF QUEUES.

⇒ Similar approach to JACKSON THEOREM's PROOF, though we are now only considering one type of transition (3) and ~~...~~ (INTERNAL MOVEMENT). ⇒ ONLY $(i \neq j)(K_j)$ INTERNAL TRANSITIONS

⇒ Because we are dealing with CLOSED MARKOVIAN NETWORKS of queues, we are only ~~interested~~ concerned with INTERNAL STATE TRANSITIONS (3) (3)

Γ_N

$\underline{N} = (k_1, \dots, k_i, \dots, k_j, \dots, k_N)$

TRANS. PROB.

RATE of SOURCE

$r_{ij}\,\alpha_j(k_i+1)\,\mu_i$
(FROM Q_i to Q_j)

$r_{ji}\,\alpha_j(k_j)\,\mu_j$
(FROM Q_j TO Q_i)

$\underline{N}_{i,j} = (k_1, \dots, k_i+1, \dots, k_j-1, \dots, k_N)$

$\boxed{\alpha_i(k_i) = \min(k_i, m_i)}$ ⇒ COEFFICIENT of the RATE

FINITE # STATES in CLOSED NETWORK (MN to K) ⇒ Always ERGODIC
⇓
Analyze STEADY-STATE!

JACKSON! (INFINITE # STATES!) ⇒ Not always Ergodic!

$\boxed{\lambda_i < m_i \mu_i}$

Γ_N:

$\boxed{N = \#\text{ QUEUES}}$

$\sum_{i=1}^{N} k_i = K$

CONSTANT

$\boxed{K = \#\text{ CUSTOMERS in the network.}}$

Γ_N: $\underbrace{P_N \sum_{j=1}^{N} \sum_{i=1}^{N} r_{ji}\,\alpha_j(k_j)\,\mu_j}_{\text{EXIT FLOW (ALL POSSIBLE TRANSITI-}}$ =

$$= \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} P_{N_{ij}} \cdot r_{ij} \, \alpha_i(k_i+1) \cdot \mu_i}_{\text{ENTERING FLOW}}$$

$$\Rightarrow \underbrace{P_{\underline{N}} \left[\sum_{j=1}^{N} \alpha_j(k_j) \mu_j \sum_{i=1}^{N} r_{ji} \right]}_{\text{EXIT FLOW}}$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} P_{N_{ij}} \cdot r_{ij} \, \alpha_i(k_i+1) \mu_i$$

PRODUCT-FORM SOLUTION

~~(STEPS to come to it!!)~~

↳ STEPS to come to a PRODUCT-FORM SOLUTION.

$$\boxed{P_{\underline{N}} = \frac{1}{G} \cdot \prod_{i=1}^{N} h_i(k_i)}$$

[of this form]

$\boxed{h_i(k_i)}$ are the expressions of the STATE PROBABILITIES of the M/M/N_S (M/M/M_i) QUEUES without $p_i(0)$

$$\Rightarrow \boxed{\sum_{\underline{N}} P_{\underline{N}} = 1}$$

NORMALIZATION CONDITION.

(Sum of all possible STATE PROBABILITIES' configurations, given that we have a fixed # CUSTOMERS (K)).

$n_1 + n_2 + \dots + n_N = k$

$k_1 + k_2 + \dots + k_N = K$

$P_{\underline{N}}$ = P. to be in a certain state $\underline{N}$

$$\triangleq \sum_{N} \prod_{i=1}^{N} h_i(k_i) = 1$$

$\hookrightarrow G$

$\underline{N}$ $\hookrightarrow$ Sum of all possible state CONFIGURATIONS PROBABILITIES

$$\Rightarrow \boxed{G = \sum_{N} \prod_{i=1}^{N} h_i(k_i)}$$

b

HOW TO OPERATE WITH THE GORDON-NEWELL THEOREM:

[CLOSED MARKOVIAN NETWORKS OF QUEUES]

(1) WRITE the ~~TRAFFIC~~ FREQUENCY EQUATIONS.

$$\boxed{\lambda_i = \sum_j \lambda_j \cdot r_{ji}} + \cancel{\gamma_i} \quad \forall i$$

THROUGHPUT of Q_j (from Q_j to Q_i) N = # QUEUES

$\hookrightarrow$ ENTERING into QUEUE i.

det = ϕ

$\Rightarrow$ We can lie down a LINEAR + HOMOGENEOUS SYSTEM of EQUATIONS, where the SPACE of solutions is given by

$$\Lambda = (1, V_2, V_3, \ldots, V_N) \qquad V_i = \frac{\lambda_i}{\lambda_1}$$

NORMALIZED COMPONENT

$\Rightarrow$ Any multiple of one solution is still a solution [fixing the topology]

(2) MULTIPLY $h_i(k_i)$ of M/M/Mi type (without $p_i(0)$). in the PRODUCT-FORM SOLUTION

[Lay out PRODUCT-FORM & the different factors]

SOLUTION

③ Evaluate $\boxed{G}$, normalizing the STATE PROBABILITIES over all the possible states of the CLOSED NETWORK of queues with K customers inside

$$\sum_{N} P_{\underline{N}} = 1$$

P_i

$$G = \sum_{N} \prod_{i=1}^{N} h_i(k_i)$$

④ Evaluate the true $\boxed{\lambda_i}$ from the MARGINAL PROBABILITIES of each queue and therein $\boxed{\Gamma_j}$ THROUGHPUT of queue

$$\Gamma_j = A_j \, \mu_j \quad \delta \; \cdot \; 1 - P\{P_j = 0\}$$

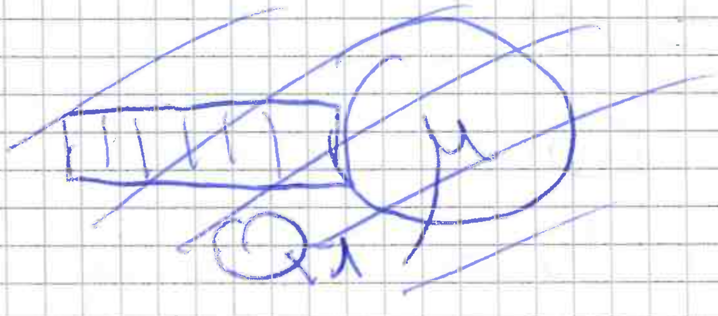
⇒ true STATE PROBABILITY of NETWORK of QUEUES

⇓

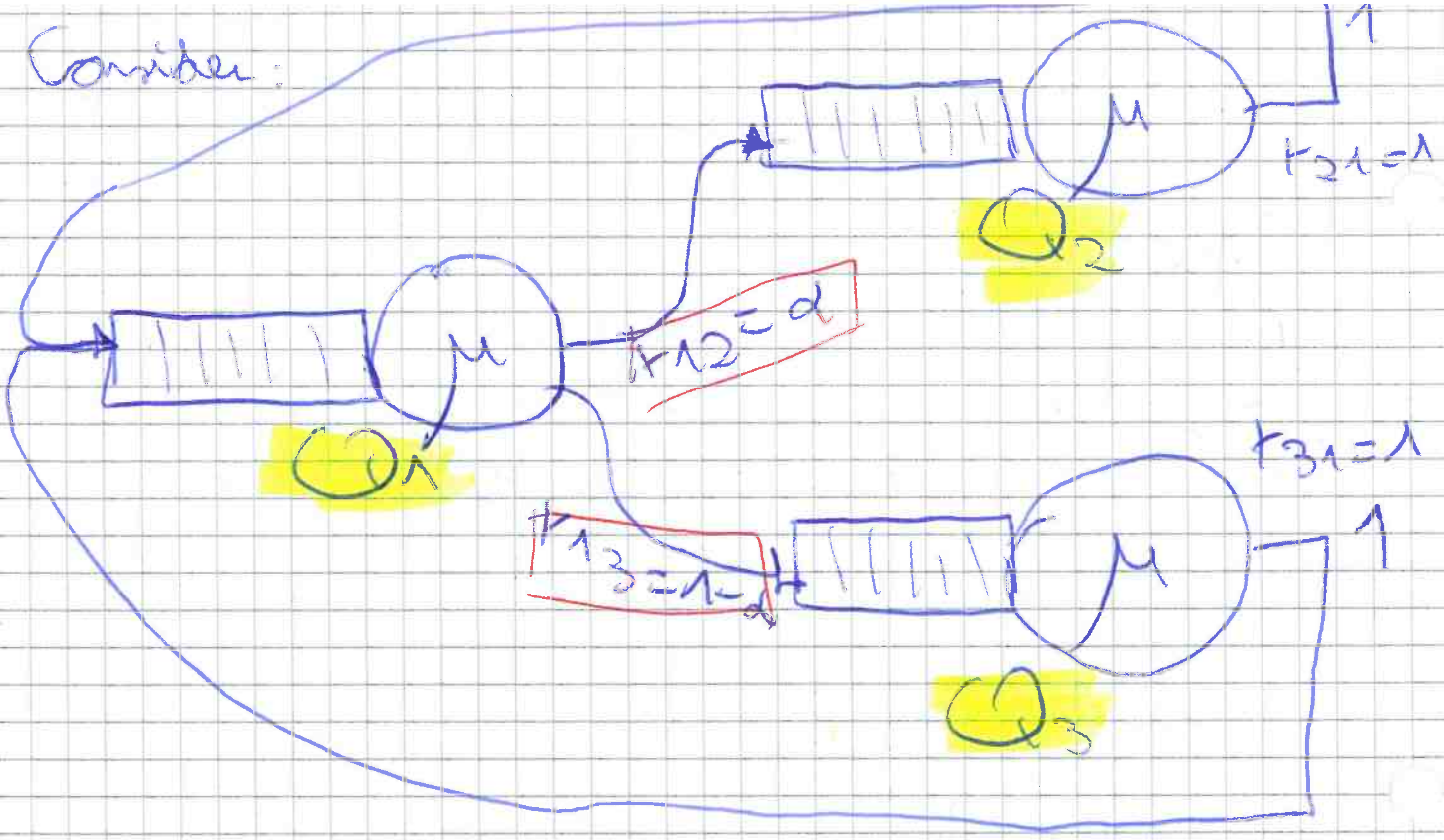
STATE PROBABILITY of each queue

ANALYSIS OF A CLOSED MARKOVIAN NETWORK OF QUEUES (USING G THE GORDON - NEWELL THEOREM)

All the servers are EXPONENTIAL!, CLOSED NETWORK of QUEUES.



Consider:



$r_{12} = \alpha \qquad r_{13} = 1-\alpha$

$r_{21} = 1 \qquad r_{31} = 1$

$\Rightarrow N_{si} = m_i = 1$ [1 SERVER in each queue]

$\alpha = \frac{1}{4}$ $\Rightarrow$ All the possible COMBINATIONS of STATES

$$\boxed{\# \text{ STATES} = N_{\text{STATES}} = \binom{Q + N_U - 1}{Q - 1}}$$

Q = # QUEUES in the network

N_U = # USERS

$\Rightarrow$ Presently: $Q = 3$ (# QUEUES)

$N_U = 3$ (3 users)

$$\Rightarrow N_{STATES} = \binom{3+3-1}{2} = \frac{5!}{2!\,3!} = \frac{5\cdot 4\cdot 3\cdot \cancel{2}}{3!\cdot\cancel{2}}$$

$$\Rightarrow N_{STATES} = \frac{5\cdot\cancel{4}^{2}\cdot\cancel{3}}{\cancel{3}\cdot\cancel{2}} = 10$$

We know we are walking with $\boxed{M|M|1 \text{ QUEUES.}}$

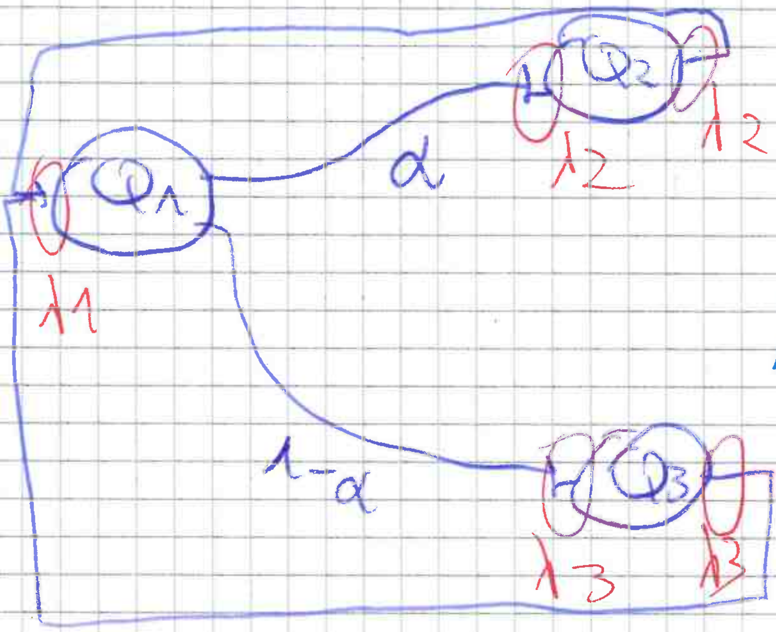
$$P_i(k_i) = (1-\rho_i)\cdot\rho_i^{k_i}$$

$$\boxed{h_i(k_i) = \left(\frac{\Lambda_i}{\mu_i}\right)^{k_i}}$$

$\Rightarrow$ The PRODUCT-FORM SOLUTION has FORM

$$\boxed{P_{k_1,k_2,k_3} = \frac{1}{G}\cdot\prod_{i=1}^{3} h_i(k_i)}$$

Where $k_1+k_2+k_3 = N_C = 3$



EXAMPLE [NO THEORY]

1) STEP: SOLVE TRAFFIC EQUATIONS:

$$\begin{cases} \lambda_2 = \alpha \lambda_1 \\ \lambda_3 = (1-\alpha)\lambda_1 \\ \lambda_1 = \lambda_2 + \lambda_3 \end{cases}$$

We could choose any other possible VALUE.

$\Rightarrow$ If we set $\boxed{\lambda_1 = 1}$ (and we know $\alpha = \frac{1}{4}$)

$\Rightarrow \lambda_2 = \alpha = \frac{1}{4} \Rightarrow \lambda_1 = \frac{\lambda_2}{\alpha} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$

$\Rightarrow \lambda_3 = \lambda_1 - \lambda_2$

$= 1 - \frac{1}{4} = \frac{3}{4}$

$$\Rightarrow \underline{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$$

VECTOR SPACE SOLUTION

$\Rightarrow$ If we set $\boxed{\lambda_1 = \lambda}$

$$\Rightarrow \lambda = \begin{pmatrix} \lambda \\ \frac{\lambda}{4} \\ \frac{3\lambda}{4} \end{pmatrix}$$

STILL A SOLUTION!

(68) BCMP NETWORK

Baskett
Chandy
Moody
Palacios-Gomez

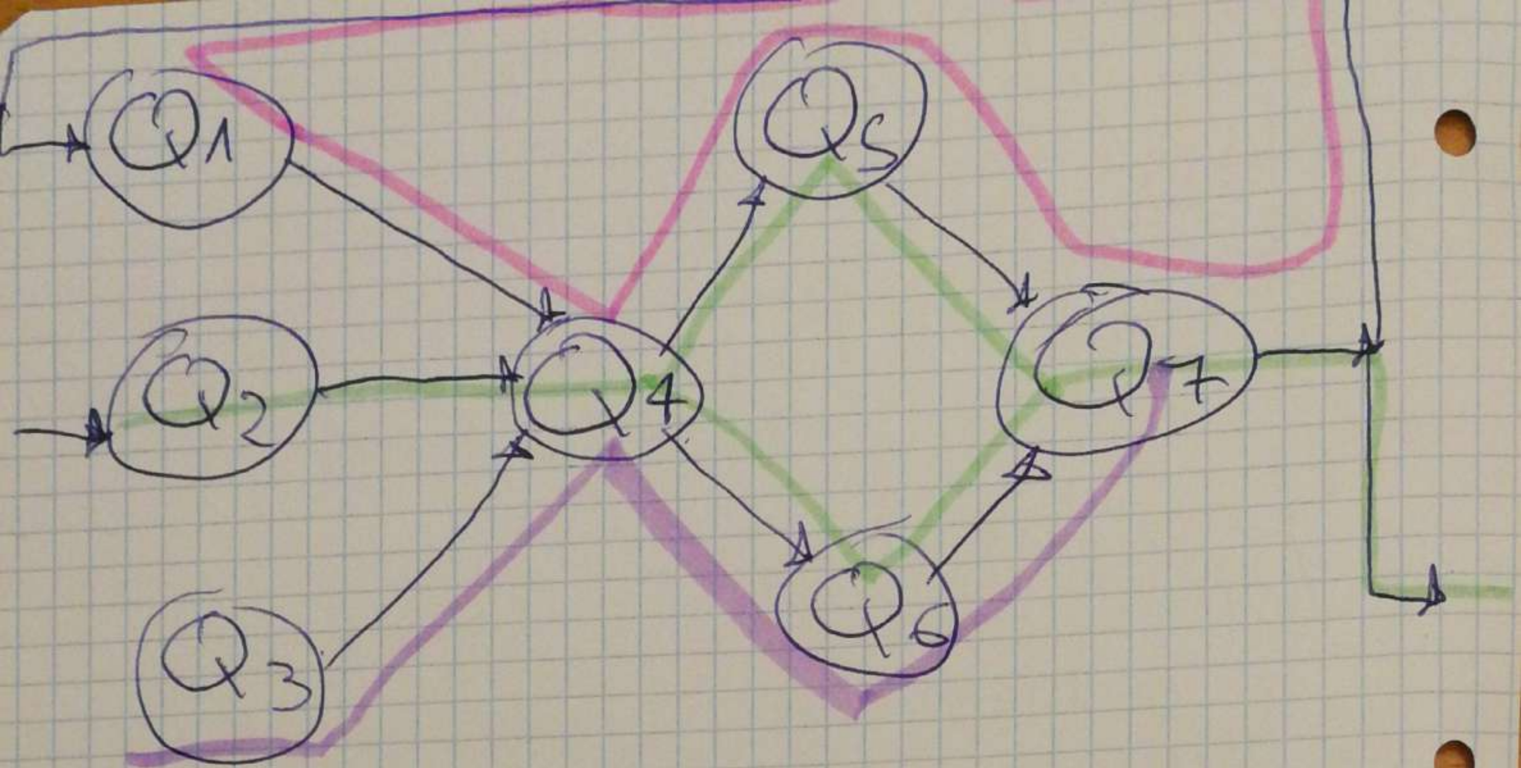
RESULT of NETWORK:
Once again, a PRODUCT-FORM SOLUTION to find the state PROBABILITY.

CHARACTERIZATION:

- Multiple classes of CUSTOMERS (i.e: different T-SHIRTS customers can wear).
- Multiple queuing disciplines. (i.e: FCFS, LCFS, PS)
- More general distribution of the SERVICE time (No longer just EXPONENTIAL DISTRIBUTION).

IMPORTANCE & USAGE:
Used for SLICING (i.e: Virtualization of a big-large device into multiple logical ones).
$\Rightarrow$ SWITCHING/CHANGE of class is possible!
[OPEN/CLOSED/HYBRID CLASS].

$\boxed{\text{M QUEUES, R CLASSES}}$ are possible.



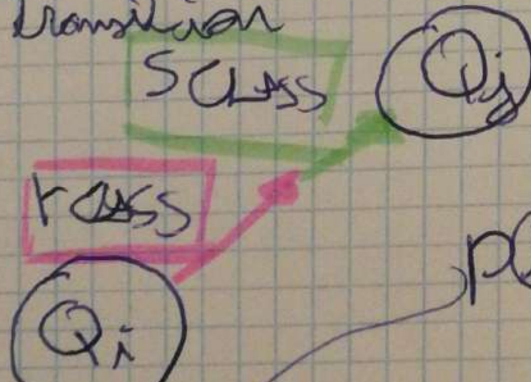
= OPEN CLASS

= SWITCHING CLASS / change T-SHIRT

= CLOSED CLASS

$\boxed{\text{ROUTING PROBABILITY}}$ (old r_{ij})

P. to transition



From QUEUE   TO QUEUE

$p_{i(r),j(s)}$

From CLASS   TO CLASS

→ Behind the routing & change of class probability, I can recognise a MARKOV CHAIN!

Generally, in fact:

$\boxed{\text{Pigits}}$ is a ~~set of~~ $\boxed{\text{REDUCIBLE MARKOV CHAIN.}}$

$\Downarrow$

We can then identify a set of IRREDUCIBLE SUBSETS of STATES (Become different classes & separate groups of STATES)

If we can assume to avoid PERIODIC STATES & that the #STATES in each subset is FINITE $\Rightarrow$ We have ERGODIC SUBSETS

(each are APERIODIC, IRREDUCIBLE, FINITE #STATES)

~~Let~~ These ERGODIC SUBSETS are then called:

$$\boxed{EC_1, EC_2, \ldots, EC_n}$$

n = #ERGODIC SUBSETS,
R = #CLASSES

If we have R classes and NO change of class: As many ergodic subsets as classes

$$n = R$$ (#ERGODIC SUBSETS = #CLASSES)

If we can change class: less groups than

$$n < R$$ (#ERGODIC SUBSETS < #CLASSES)

[Aggregate some subsets]

For each queue, we can now define a new state $\underline{N}_i$ (w.r.t. to the different classes).

$$\boxed{\underline{N}_i = (N_{i1}, N_{i2}, \dots, N_{iR})}$$

STATE of QUEUE i (one ROW)

Customers of Class 1 (# customers of class 1 in queue i)

Class 2

Class R

$\underline{\underline{N}}$ = MATRIX of the STATE of the SYSTEM (not a vector) (#CUSTOMERS of class j in queue i)

STATE CHARACTERIZATION: #CUSTOMERS of class j in Queue i

$$\boxed{N_{i,j}}$$

TYPES of USER CLASSES for BCMP NETWORKS of QUEUES:

- OPEN
- CLOSED
- MIXED

If some classes are CLOSED, then for them

#QUEUES

$$\sum_{i=1}^{M} \sum_{r \in CQ} N_{i,r} = NQ$$

CQ: Subsets corresponding to CLOSED CLASSES

(Constant #CUSTOMERS in the closed classes).

BCMP ANALYSIS:

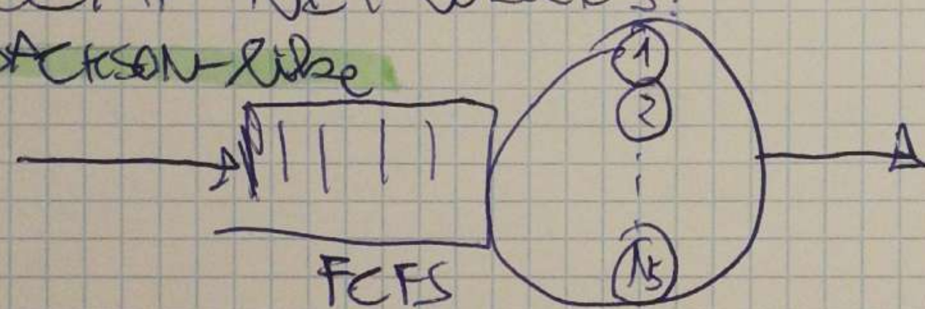
— EXTERNAL ARRIVALS:

- Poissonian with FIXED RATE (UDP | Fixed-rate UDP)
- Poissonian with RATE depending on the STATE of the QUEUE.

Ex: TCP - Changes packet transmission rate based on the network bandwidth.

TYPES of QUEUES we can use in BCMP NETWORKS:

① JACKSON-like

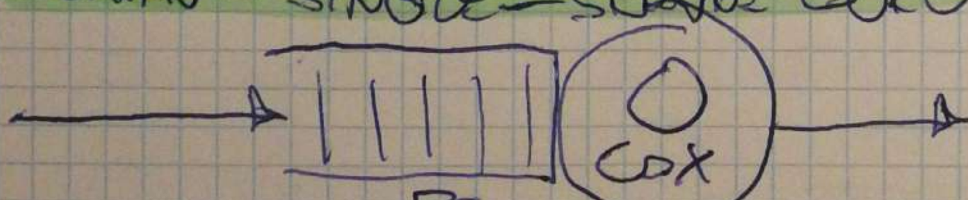


SERVING POLICY

- FCFS • $N_S = \infty$ • MULTIPLE SERVERS
- EXP. DISTRIBUTION of SERVICE TIME
- SAME $E\{\tau_S\}$ for ALL CLASSES
- SERVICE TIME depends on the # CUSTOMERS in the queue. (adaptive)

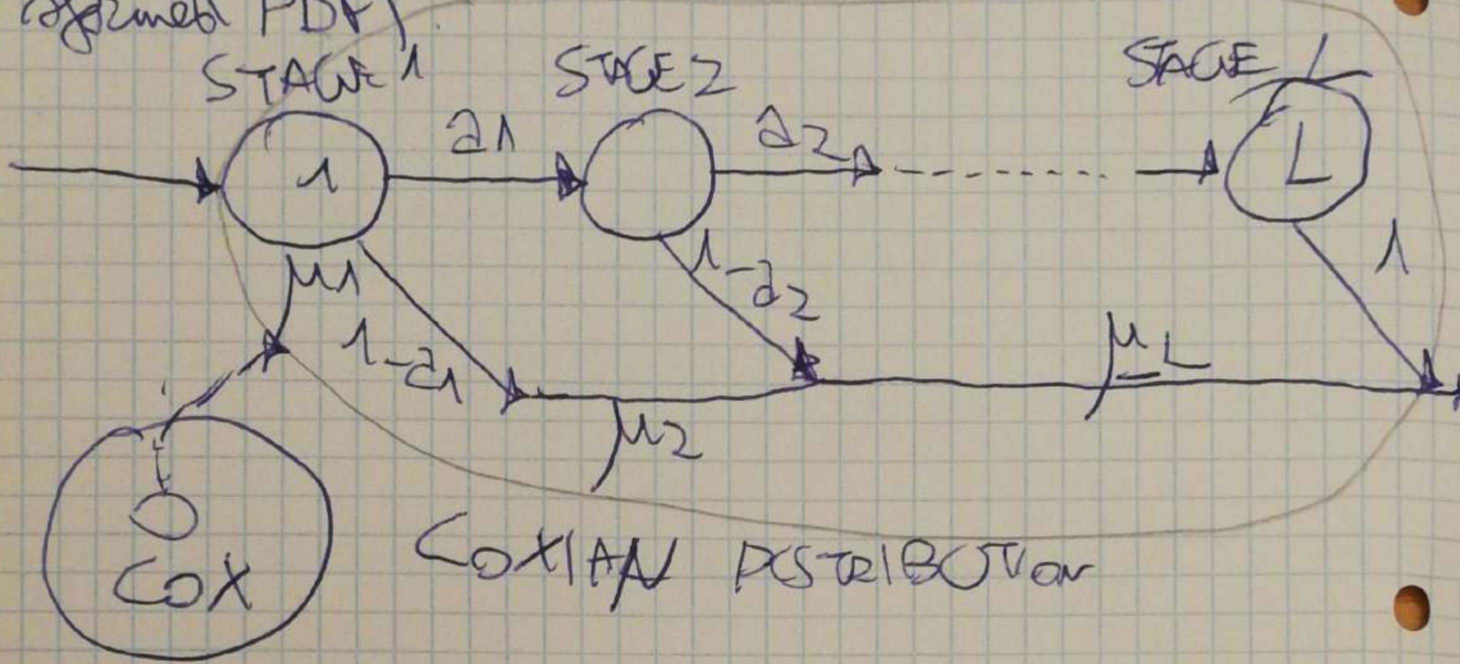
⇒ Increasing # CUSTOMERS ⇒ Server's behaviour changes & can be analysed.

② COXIAN - SINGLE - SERVER QUEUE (IDEAL)



- PS SERVING POLICY (Processor Sharing)

COXIAN = Very General PDF
(any PDF having a fractional form of its transformed PDF)

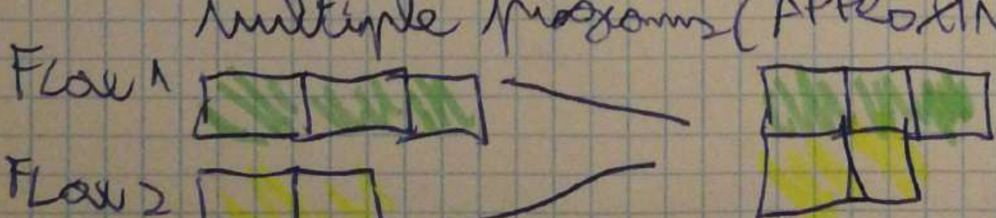


COXIAN DISTRIBUTION

- $N_S = 1$ (single-server)
- DIFFERENT for each CLASS
- COXIAN DISTRIBUTION
- PS - QUEUEING DISCIPLINE

PS: Ideally, want to have simultaneous flows.
(Actually)

→ ROUND-ROBIN over Δt
(Infinitesimal service time). ↳ $\Delta t \to 0$
↳ SIMULATE the simultaneous execution of multiple programs (APPROXIMATION).

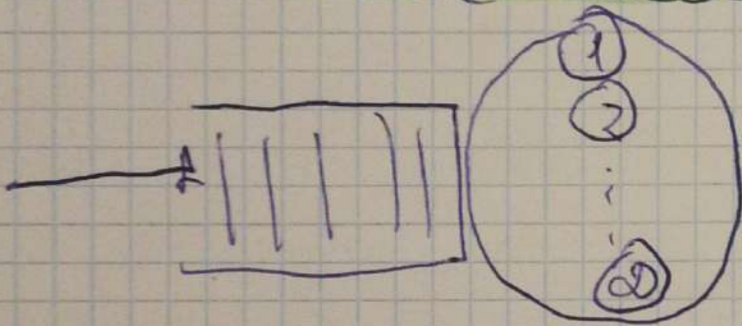


(3) COX-LCFS

LCFS

- $N_S = 1$ • COXIAN DISTRIBUTION
- DIFFERENT for each class
- LCFS queueing discipline

(4) INFINITE SERVER QUEUE (DELAY MODEL)



- $N_S = \infty$ • COXIAN DISTRIBUTION

($E\{N\} = \infty$)

(Used to emulate the processing delay in a PROCESSOR | TRANSMISSION DELAY in a system).

DETERMINISTIC

BCMP NETWORK'S IDEA.

STATE of the NETWORK = Vector of vectors (MATRIX)

(simple definition)

↳ # customers in each queue for each class.

BCMP NETWORK'S THEOREM.

⇓

PRODUCT-FORM SOLUTION.

$$\boxed{P(\underline{N}) = \frac{1}{G} \cdot \prod_{i=1}^{N} g_i(N_i)}$$

$\underline{N}$ → MATRIX

N → # QUEUES

N_i → VECTOR

G → NORMALIZATION COEFFICIENT

How to OPERATE with the BCMP NETWORKS.

1. Solve the TRAFFIC EQUATIONS.

$$\boxed{\lambda_{i,r} = \gamma_{i,r} + \sum_{j=1}^{M} \sum_{s=1}^{Q} \lambda_{j,s} \cdot P_{j,s,i,r}}$$

Q = #

$\gamma_{i,r}$ → if OPEN NETWORK (arrivals from the external)

$i = 1, \dots, M$ # QUEUES

$r = 1, \dots, Q$ # user classes

2. Check the ERGODICITY CONDITION.

$$\sum_{r=1}^{Q_0} \frac{\lambda^*_{i,r}}{\mu^*_{i,r}} < 1$$

Q_0 → OPEN CLASSES

$\lambda_{i,r}$, $\mu_{i,r}$ → Dependent on the # CUSTOMERS in the NETWORK

$$\lambda^*_{i,r} = \lim_{j \to +\infty} \lambda_{i,r}(j)$$

$$\mu^*_{i,r} = \lim_{j \to +\infty} \mu_{i,r}(j)$$

NOT dependent on the # CUSTOMERS in the network

In case of BCMP NETWORKS, where:

1. USERS cannot change classes
2. FREQUENCIES λ_i, μ_i do not depend on the # CUSTOMERS in the network.

$$
g_i(N_i) =
\begin{cases}
N_i! \cdot \prod_{r=1}^{R} \left(\frac{1}{N_{i,r}!}\right)\left(\frac{V_{i,r}}{\mu_i}\right)^{N_{i,r}} & \text{"JACKSON-LIKE" } (1) \\[2ex]
\frac{(N_i)!}{\mu_i^{N_i}} \prod_{r=1}^{R} \frac{V_{i,r}^{N_{i,r}}}{N_{i,r}!} & (1) \\[2ex]
N_i! \cdot \prod_{r=1}^{R} \left(\frac{1}{N_{i,r}!}\right)\left(\frac{V_{i,r}}{\mu_{i,r}}\right)^{N_{i,r}} & \text{PS-COXIAN } (2),\ \text{LCFS-COXIAN } (3) \\[2ex]
\prod_{r=1}^{R} \frac{1}{N_{i,r}!} \cdot \left(\frac{V_{i,r}}{\mu_{i,r}}\right)^{N_{i,r}} & \text{NS}=\infty \text{ COXIAN } (4)
\end{cases}
$$

$$
V_{i,r} =
\begin{cases}
\frac{\lambda_{i,r}}{\lambda_{1,r}} & \text{if } r \text{ is CLOSED CLASS} \\[1ex]
\lambda_{i,r} & \text{if } r \text{ is OPEN CLASS}
\end{cases}
$$

$$
N_i = \sum_{r=1}^{R} N_{i,r}
$$

(NORMALIZATION CONDITION over all classes of one queue i)

(2) Write the $\boxed{\text{PRODUCT FORM}}$

$$P_{k_1 k_2 k_3} = \frac{1}{G} \cdot \left(\frac{\lambda_1}{\mu_1}\right)^{k_1} \left(\frac{\lambda_2}{\mu_2}\right)^{k_2} \left(\frac{\lambda_3}{\mu_3}\right)^{k_3}$$

SUBSTITUTE OBTAINED VALUES with the TRAFFIC EQUATIONS

$$= \frac{1}{G} \cdot \left(\frac{\lambda_1}{\mu}\right)^{k_1} \cdot \left(\frac{\lambda_1}{4\mu}\right)^{k_2} \cdot \left(\frac{3\lambda_1}{4\mu}\right)^{k_3}$$

$$= \frac{1}{G} \cdot \left(\frac{\lambda_1}{\mu}\right)^{k_1+k_2+k_3} \quad \cancel{\frac{3}{4}} \Rightarrow \left(\frac{\lambda_1}{\mu}\right)^{k_1+k_2+k_3} = \left(\frac{\lambda_1}{\mu}\right)^{3}$$

$$= \frac{1}{G'} \cdot \left(\frac{1}{1}\right)^{k_1} \cdot \left(\frac{1}{4}\right)^{k_2} \cdot \left(\frac{3}{4}\right)^{k_3}$$

[we know]

(3) Evaluate $\boxed{G'}$ (G) ⇒ EXHAUSTIVE ANALYSIS of all possible STATES

We found that: $N_{STATES} \binom{Q+N_C-1}{Q-1} = 10$

~~⇒ Exhausti~~

⇓

All possible states are, with $\boxed{N_C = 3}$

(3,0,0); (0,3,0); (0,0,3); (2,1,0); (0,1,2)
(2,0,1); (0,2,1); (1,1,1); (1,0,2); (1,2,0)

$$P_{k_1,k_2,k_3} = k \cdot 1^{k_1} \cdot \left(\frac{1}{4}\right)^{k_2} \cdot \left(\frac{3}{4}\right)^{k_3}$$

$P(3,0,0) = (1)^3 = 1 \cdot k$

$P(0,3,0) = \left(\frac{1}{4}\right)^3 \cdot k$

$P(0,0,3) = \left(\frac{3}{4}\right)^3 \cdot k$

$P(2,1,0) = \frac{1}{4}\, k$

$P(2,0,1) = \frac{3}{4}\, k$

$P(1,2,0) = \left(\frac{1}{4}\right)^2 k$

$P(0,2,1) = \left(\frac{1}{4}\right)^2 \cdot \frac{3}{4}\, k$

$P(1,0,2) = \left(\frac{3}{4}\right)^2 k$

$P(0,1,2) = \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 k$

$P(1,1,1) = \frac{1}{4} \cdot \left(\frac{3}{4}\right) k$

$\textcircled{+} = \frac{55}{16} k = 1$

Σ

$\Rightarrow \boxed{k = \frac{16}{55}} = \boxed{\frac{1}{G}}$

NB: We know the NORMALIZATION CONDITION

$$\boxed{\sum_{k_1+k_2+k_3} P_{k_1 k_2 k_3} = 1}$$

$\Rightarrow$ We sum all the MARGINAL PROBABILITIES and set the Result $= 1$

~~~~ We have that

$$\cancel{k = \frac{55}{16} = \frac{1}{G}} \quad k = \frac{16}{55} = \frac{1}{G}$$

~~We have hence found that~~

And knowing that the PRODUCT FORM SOLUTION is:

$$\boxed{P_{k_1 k_2 k_3} = \frac{1}{G} \cdot (1)^{k_1} \cdot \left(\frac{1}{4}\right)^{k_2} \cdot \left(\frac{3}{4}\right)^{k_3}}$$

$$\Rightarrow P_{k_1 k_2 k_3} = \frac{16}{55} \cdot (1)^{k_1} \cdot \left(\frac{1}{4}\right)^{k_2} \cdot \left(\frac{3}{4}\right)^{k_3}$$

$$\boxed{P_{k_1 k_2 k_3} = \frac{16}{55} \cdot \left(\frac{1}{4}\right)^{k_2} \cdot \left(\frac{3}{4}\right)^{k_3}}$$

(4) Find the REAL (TRUE) λ_i from the STATE PROBABILITIES of the NETWORK, we can then evaluate the MARGINAL STATE PROBABILITIES of each QUEUE.

MARGINAL STATE PROBABILITY $\boxed{\pi_{i,j}} \Rightarrow$ P(j customers in Q_i)

$\Rightarrow$ PROBABILITIES of a certain SUBSET

(~~Evaluate~~ Sum all the different POSSIBILITIES to have such situation).

$$\pi_{1,0} = P_{0,3,0} + P_{0,0,3} + P(0,2,1) + P(0,1,2)$$

$$\pi_{1,1} = P_{1,2,0} + P_{1,0,2} + P_{1,1,1}$$
~~~~

$\pi_{1,2} = p_{2,1,0} + p_{2,0,1}$

$\pi_{1,3} = p_{3,0,0}$

AVERAGE # CUSTOMERS IN QUEUE 1:

If we have the marginal PROBABILITIES, we can evaluate ~~evaluate~~ $\boxed{E\{n_i\}}$

$$\boxed{E\{n_1\} = \sum_{i=1}^{3} i \cdot \pi_{1,i}} \quad \left[\text{as } E\{n\} = \sum_{n=0}^{\infty} n \cdot p_n\right]$$

$$\Rightarrow E\{n_1\} = \sum_{i=1}^{3} i \cdot \pi_i = 1 \cdot \pi_{1,1} + 2\pi_{2,1} + 3\pi_{3,1}$$

THROUGHPUT of QUEUE (1)

$$\boxed{\Gamma_1 = \sum_{i=1}^{3} \mu_i \cdot \pi_{1,i}} = \sum_{i=1}^{3} \mu \cdot \pi_i$$

n-th QUEUE

$\mu_i = \mu$ [BECAUSE single SERVER QUEUE]

$$\Gamma_1 = \mu \cdot [\pi_{1,1} + \pi_{1,2} + \pi_{1,3}]$$
$$= \mu [1 - \pi_{1,0}]$$

All cases with at least one ~~customer~~ in the QUEUE waiting for the ~~the~~ RE.

$\Rightarrow$ For M/M/1:

$$\Gamma_1 = \lambda_1$$

If you have λ_1, you have all others too!

NB: $\boxed{E\{n_1\} + E\{n_2\} + E\{n_3\} = N_U}$, $N_U = 3$

~~$E\{n_1\} = 8$~~

LECTURE 12

4) ⇒ EVALUATION of TRUE λ_i from the THROUGHPUT of each QUEUE.

We found (for M/M/1 QUEUE):

FREQUENCY of ACCEPTED CUSTOMERS

$$P\{\pi_1 \cdot p_2 \pi_3 = \pi_i k\} \Rightarrow \boxed{\Gamma_i = \Lambda_i} = \lambda_i$$

$$\Gamma_{1} = \mu(1 - \pi_{1,0}) = \lambda_1$$

$$\Gamma_1 = \mu(1 - p_{030} - p_{003} - p_{021} - p_{012})$$
$$= \mu \frac{9}{11}$$

$$\Rightarrow \Gamma_1 = \mu \cdot \frac{9}{11} = \Lambda_1 = \lambda_1$$

We know that: $\boxed{\lambda_1 = \frac{9}{11}\mu}$ ⇒ Substituting in $\lambda_1 = \frac{9}{11}\mu$

$$\boxed{\underline{\lambda} = \begin{pmatrix} \lambda_1 \\ \frac{1}{4}\lambda_1 \\ \frac{3}{4}\lambda_1 \end{pmatrix}} = \begin{pmatrix} \frac{9}{11}\mu \\ \frac{9}{44}\mu \\ \frac{27}{44}\mu \end{pmatrix}$$

⇒ We can then evaluate the AVERAGE #CUSTOMERS based on the MARGINAL PROBABILITIES

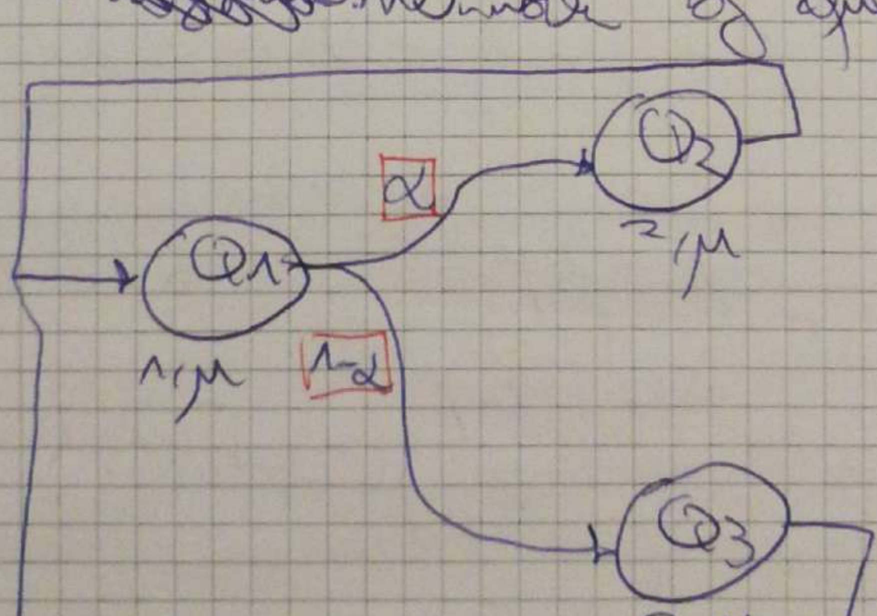
$$E\{n_1\} = \sum_{i=1}^{3} i\,\pi_{1i} = 1\cdot\pi_{11} + 2\pi_{12} + 3\pi_{13}$$

$$\Rightarrow \boxed{E\{n_j\} = \sum_{i=1}^{+\infty} i\cdot\pi_{ji}}$$

AVERAGE TIME SPENT IN EACH QUEUE

{IN A CLOSED NETWORK OF QUEUES}

$\Rightarrow$ We want to apply LITTLE'S THEOREM to our ~~system~~ network of queues.



$$\Rightarrow \boxed{E\{n\} = \Lambda\cdot E\{\tau\}}$$

$$\Rightarrow \boxed{E\{\tau_i\} = \frac{E\{n_i\}}{\Lambda_i}}$$

AVG. TIME SPENT IN "CROSSING" QUEUE i

ALTERNATIVELY:

Use Little's Formula.

$$\boxed{E\{\tau\}\Lambda = E\{N\}} = 3$$

$$\Rightarrow E\{\tau\} = \frac{3}{\Lambda_1} = \frac{3}{\lambda_1} = \frac{3}{\frac{9}{11}\mu} = \frac{11}{3}\cdot\frac{1}{\mu}$$

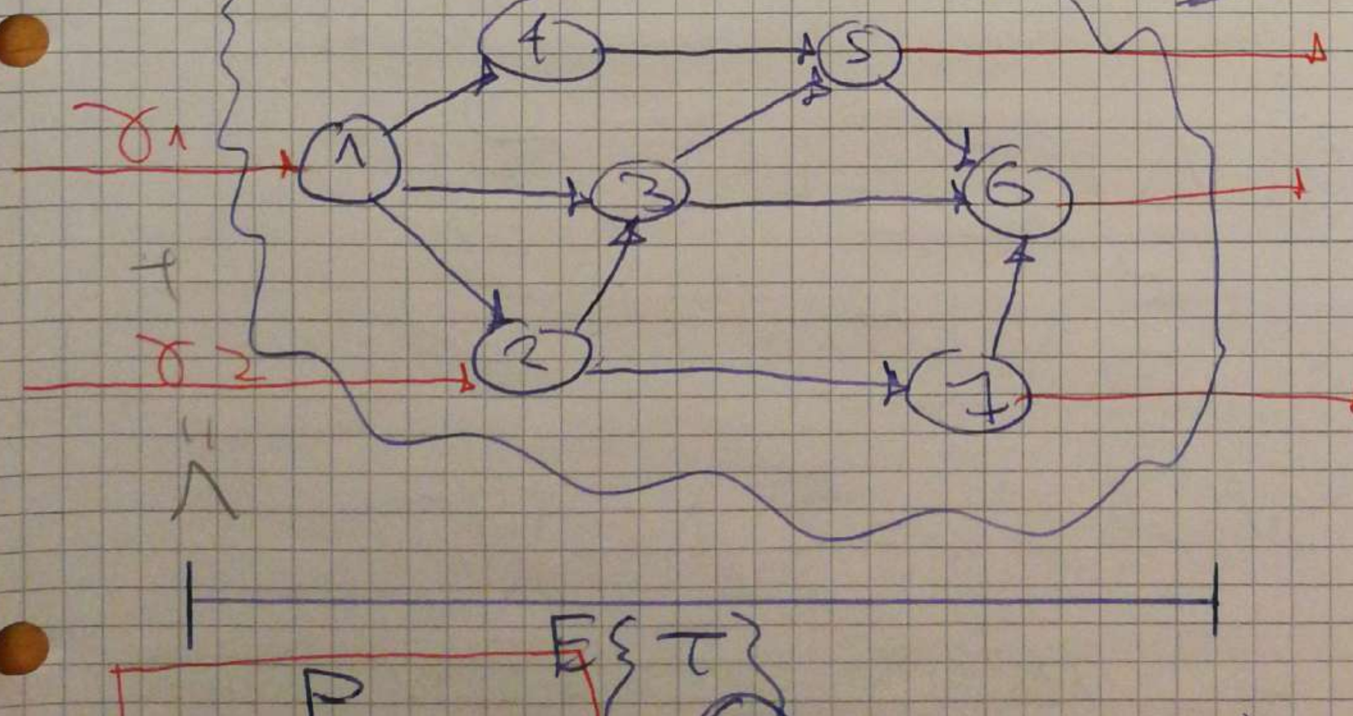
G7

TRANSIT TIME IN AN OPEN NETWORK OF QUEUES

Our course is specifically aimed at evaluating the CROSSING TIME (TRANSIT) of OPEN NETWORK of QUEUES / A ROUTER.

$$\boxed{E\{n\} = \Lambda \cdot E\{\tau\}}$$

$\cancel{\Lambda = \sum_{i=1}^{P} \gamma_i}$



$$\boxed{\Lambda = \sum_{i=1}^{P} \gamma_i}$$

P = # QUEUES with an INPUT from the EXTERNAL.

$$\boxed{E\{n\} = \Lambda \cdot E\{\tau\}}$$

$$\boxed{\Lambda = \sum_{i=1}^{P} \gamma_i}$$ P = # QUEUES with EXTERNAL INPUT.

ALSO: TOTAL # CUSTOMERS in NETWORK (ALL QUEUES) ‖

Sum of # CUSTOMERS in each QUEUE (i) of the NETWORK.

$$\boxed{E\{n\} = \sum_{i=1}^{Q} E\{n_i\}}$$

CUSTOMERS in QUEUE i:

$$\boxed{E\{n_i\} = \underbrace{\Lambda_i}_{= \lambda_i} \cdot E\{\tau_i\}}$$

⇒ Put these 2 things together:

$$\boxed{\Lambda \cdot E\{\tau\}} = E\{n\} = \sum_{i=1}^{Q} E\{n_i\}$$

$$= \boxed{\sum_{i=1}^{Q} \lambda_i \cdot E\{\tau_i\}}$$

$$\Rightarrow \Lambda \cdot E\{\tau\} = \sum_{i=1}^{Q} \lambda_i \cdot E\{\tau_i\}$$

⇒ AVERAGE TRAVERSAL TIME:

$$\boxed{E\{\tau\} = \sum_{i=1}^{Q} \frac{\lambda_i}{\Lambda} \cdot E\{\tau_i\}}$$

AVERAGE TRAVERSAL TIME:

$$\boxed{E\{\tau\} = \sum_{i=1}^{Q} \frac{\lambda_i}{\Lambda} E\{\tau_i\}}$$

Time spent in ith queue

$E\{\eta_i\}$ = # transits through Q_i.

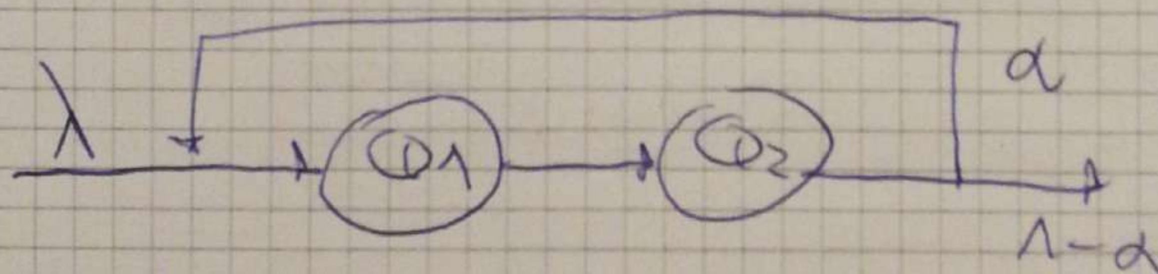
COEFFICIENT corresponding to the # TIMES of passing through | visiting the QUEUE

$$\Lambda = \sum_{i=1}^{P} \gamma_i$$

(coming from external.

EXAMPLES OF TOPOLOGIES OF OPEN NETWORKS of QUEUES.

CASE 1: TOPOLOGICAL APPROACH.



For $\alpha < 1$, $\underbrace{E\{\tau_1\}}_{\text{QUEUING TIME } Q_1}$, $\underbrace{E\{\tau_2\}}_{\text{QUEUEING TIME } Q_2}$

$$P\{n \text{ transits through } Q_1 \,\&\, Q_2\} = \begin{cases} (1-\alpha) & n=1 \\ \alpha\cdot(1-\alpha) & n=2 \\ \alpha^2\cdot(1-\alpha) & n=3 \end{cases}$$

$$\Rightarrow \boxed{P\{n \text{ transits}\} = \alpha^{n-1}(1-\alpha)} \quad n \geq 1$$

$$E\{\#\text{TRANSITS}\} = \sum_{i=1}^{+\infty} i \cdot (1-\alpha) \cdot (\alpha)^{i-1}$$

$$\Rightarrow E\{\#\text{TRANSITS}\} = \cancel{(1-\alpha)} \frac{1}{(1-\alpha)^{\cancel{2}}} = \frac{1}{(1-\alpha)}$$

TO Q1, Q2

~~ALTERNATIVELY~~

$$\Rightarrow E\{T\} = \frac{1}{1-\alpha} \cdot E\{T_1\} + \frac{1}{1-\alpha} \cdot E\{T_2\}$$

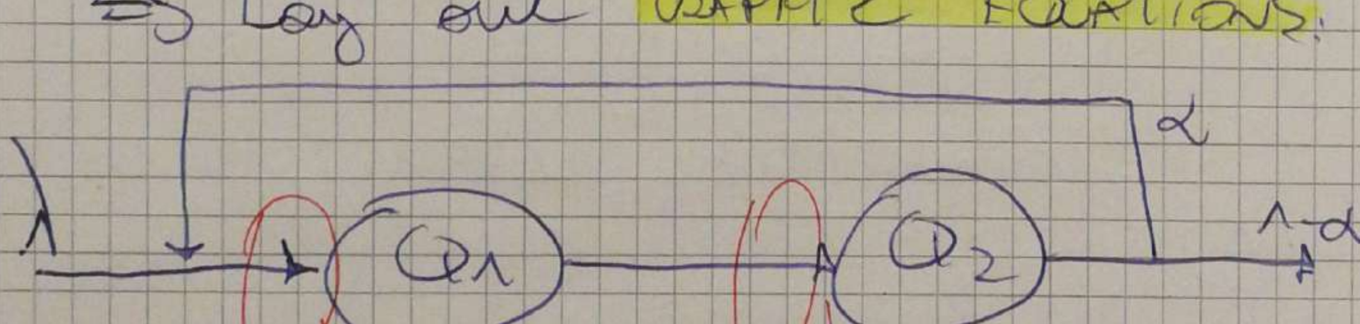
$$E\{T\} = \frac{E\{T_1\} + E\{T_2\}}{(1-\alpha)}$$

ALTERNATIVE APPROACH (TRAFFIC EQUATIONS)

$\Rightarrow$ want to find λ_1, λ_2 for:

$$\boxed{E\{T\} = \sum_{i=1}^{+\infty} \frac{\lambda_i}{\Lambda} E\{T_i\}}$$

$\Rightarrow$ Lay out TRAFFIC EQUATIONS:



$$\begin{cases} \lambda_1 = \lambda_2 \\ \underbrace{\lambda}_{\text{GLOBAL INPUT}} = \underbrace{\lambda_2 (1-\alpha)}_{\text{GLOBAL OUTPUT}} \end{cases} \Rightarrow \lambda_1 = \frac{\lambda}{(1-\alpha)} \quad \lambda_2 = \frac{\lambda}{(1-\alpha)}$$

$$E\{T\} = \frac{\lambda_1}{\Lambda} \cdot E\{T_1\} + \frac{\lambda_2}{\Lambda} \cdot E\{T_2\}$$

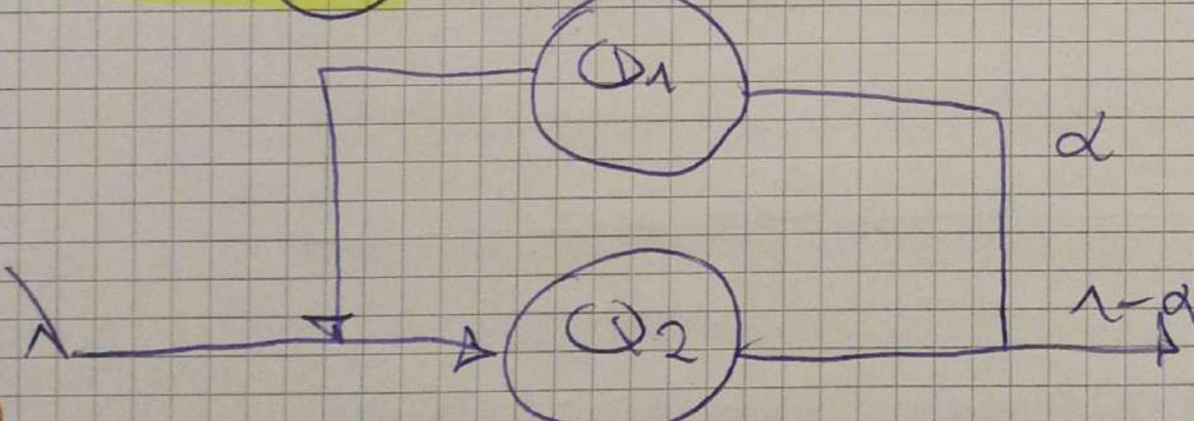
λ (what goes in) $= \Lambda$

$$= \frac{\lambda_1}{\Lambda} \cdot E\{T_1\} + \frac{\lambda_2}{\lambda} E\{T_2\}$$

$$= \frac{\Lambda}{1-\alpha} \cdot E\{T_1\} + \frac{\Lambda}{1-\alpha} \cdot E\{T_2\}$$

$$= \frac{E\{T_1\} + E\{T_2\}}{(1-\alpha)}$$

CASE 2



$\alpha < 1$ [ERGODIC] , $E\{T_1\}$, $E\{T_2\}$

$P\{n \text{ transits through } Q_1 \,\&\, Q_2\} = (\alpha)^{n-1} (1-\alpha)$

$E\{\# \text{ transits through } Q_2\} = \frac{\Lambda}{1-\alpha}$

(same situation as Λ) $E\{\eta_2\}$

$E\{\# \text{ transits through } Q_1\} = \frac{\Lambda}{1-\alpha} - \Lambda$

$$= \frac{\alpha}{1-\alpha}$$

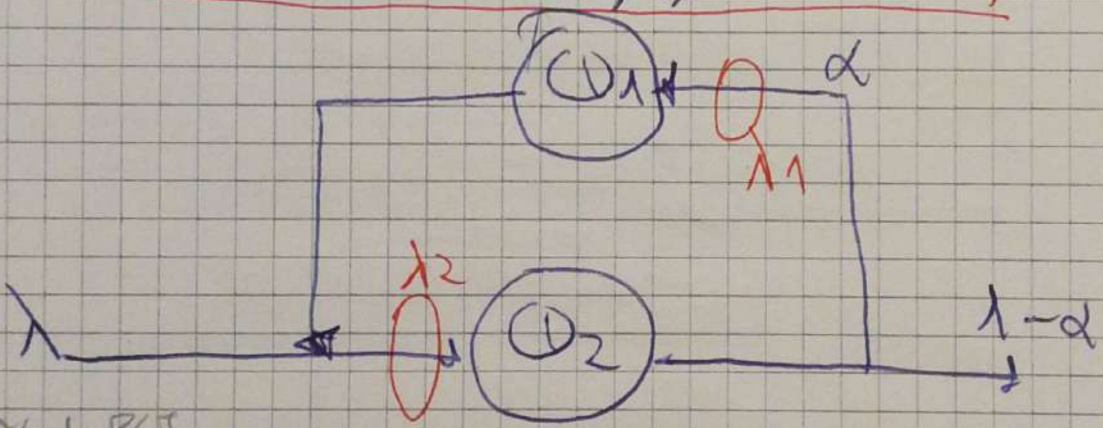
$$\Rightarrow E\{\tau\} = \frac{\alpha}{1-\alpha} E\{\tau_1\} + \frac{E\{\tau_2\}}{1-\alpha}$$

$$= \frac{\alpha E\{\tau_1\} + E\{\tau_2\}}{(1-\alpha)}$$

ALTERNATIVE APPROACH: $\Rightarrow$

Write the TRAFFIC EQUATIONS

$$\boxed{E\{\tau\} = \sum_{i=1}^{Q} \frac{\lambda_i}{\Lambda_i} E\{\tau_i\}}$$



$$\underbrace{\lambda}_{\text{GLOBAL INPUT}} = \underbrace{\lambda_2 \cdot (1-\alpha)}_{\text{GLOBAL OUTPUT}} \Rightarrow \lambda_2 = \frac{\lambda}{1-\alpha}$$

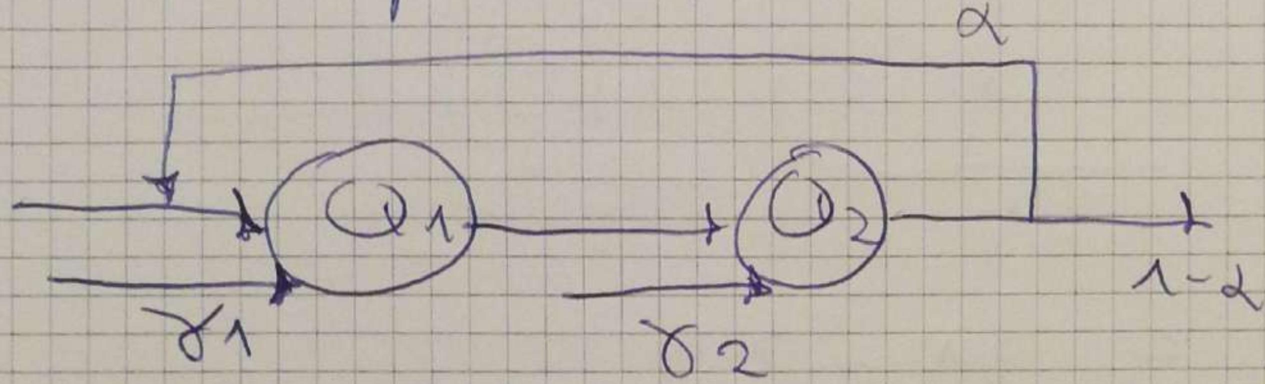
$$\lambda_1 = \alpha \lambda_2 \qquad \lambda_1 = \frac{\alpha \lambda}{1-\alpha}$$

$$\Rightarrow E\{\tau\} = \frac{\alpha}{1-\alpha} E\{\tau_1\} + \frac{1}{1-\alpha} E\{\tau_2\}$$

$$= \frac{\alpha E\{\tau_1\} + E\{\tau_2\}}{1-\alpha}$$

CASE (3) $\Lambda = \gamma_1 + \gamma_2$

Customers can enter from input 1 & from input 2 (γ_1, γ_2)



INPUT 1 (γ_1) ⇒ CASE (1)

INPUT 2 (γ_2) ⇒ CASE (2)

(1) $= E\{T\} = \dfrac{E\{T_1\} + E\{T_2\}}{1-\alpha}$ $\left(\dfrac{\gamma_1}{\gamma_1+\gamma_2}\right)$

(2) $= E\{T\} = \dfrac{\alpha E\{T_1\} + E\{T_2\}}{1-\alpha}$ $\left(\dfrac{\gamma_2}{\gamma_1+\gamma_2}\right)$

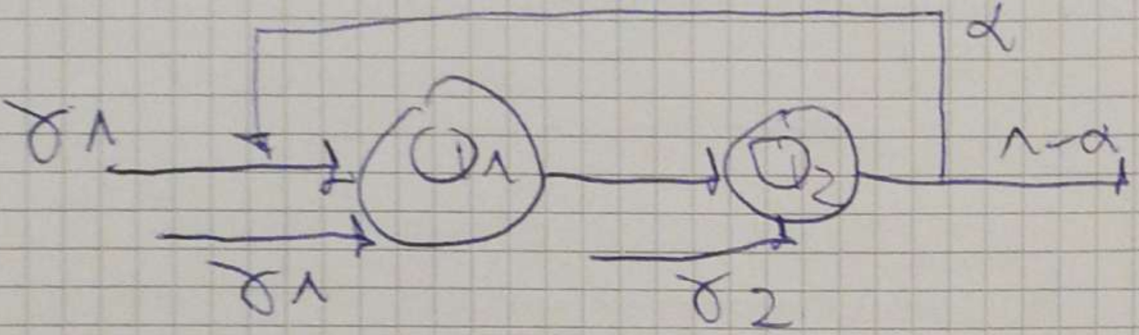
⇒ We now need to WEIGHT the two different CASES!

$$\Rightarrow \boxed{E\{T\} = \frac{\gamma_1}{\gamma_1+\gamma_2}\,\frac{E\{T_1\}+E\{T_2\}}{(1-\alpha)} + \frac{\gamma_2}{\gamma_1+\gamma_2}\,\frac{\alpha E\{T_1\}+E\{T_2\}}{1-\alpha}}$$

$$\Rightarrow E\{T\} = \frac{E\{T_1\}\cdot(\gamma_1+\alpha\gamma_2)}{(1-\alpha)(\gamma_1+\gamma_2)} + \frac{E\{T_2\}\cancel{(\gamma_1+\gamma_2)}}{(1-\alpha)\cdot\cancel{(\gamma_1+\gamma_2)}}$$

ALTERNATING APPROACH

⇒ lay down TRAFFIC EQUATIONS



$$\Lambda = \gamma_1 + \gamma_2$$

$$\left\{\begin{array}{l} \underbrace{\gamma_1 + \gamma_2}_{\text{GLOBAL INPUT}} = \underbrace{\lambda_2(1-\alpha)}_{\text{GLOBAL OUTPUT}} \\ \lambda_1 = \gamma_1 + \alpha\lambda_2 \end{array}\right.$$

$\lambda_2 = \lambda_1 + \gamma_2$ (ALTERNATIVE APPROACH)

$$\Rightarrow \boxed{\lambda_2 = \frac{\gamma_1 + \gamma_2}{(1-\alpha)}}$$

$$\Rightarrow \lambda_1 = \gamma_1 + \alpha \cdot \frac{(\gamma_1 + \gamma_2)}{(1-\alpha)}$$

$$= \frac{\gamma_1 (1-\alpha) + \alpha\gamma_1 + \alpha\gamma_2}{1-\alpha}$$

$$= \frac{\gamma_1 - \cancel{\alpha\gamma_1} + \cancel{\alpha\gamma_1} + \alpha\gamma_2}{1-\alpha}$$

$$\boxed{\lambda_1 = \frac{\gamma_1 + \alpha\gamma_2}{(1-\alpha)}}$$

$$\cancel{\Rightarrow \text{DEG} \ldots = \frac{\gamma_1 + \alpha\gamma_2}{(1-\alpha)\cdot(\gamma_1 + \gamma_2)}}$$

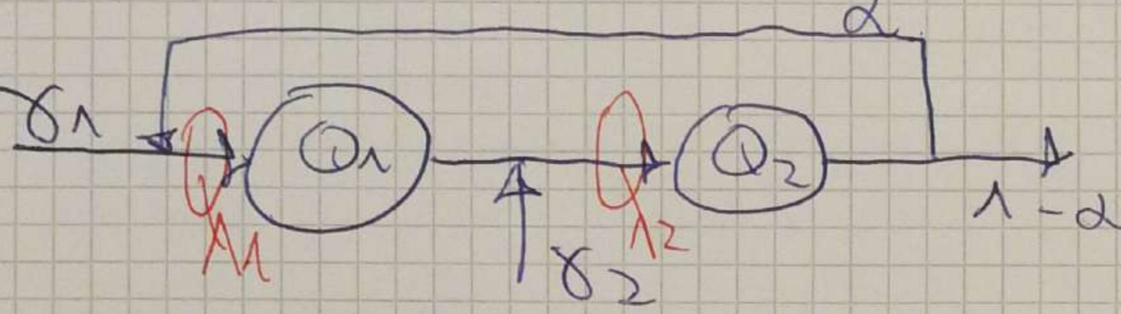
$$E\{\tau\} = \sum_{i=1}^{2} \frac{\lambda_i}{\Lambda} \cdot E\{\tau_i\}$$

$$= \frac{\lambda_1}{\Lambda} \cdot E\{\tau_1\} + \frac{\lambda_2}{\Lambda} \cdot E\{\tau_2\}$$

$$= \frac{\gamma_1 + \alpha\gamma_2}{(1-\alpha)\cdot(\gamma_1+\gamma_2)} E\{\tau_1\} + \cancel{\frac{(\gamma_1+\gamma_2)}{(1-\alpha)\cdot(\gamma_1+\gamma_2)} \cdot E\{\tau_2\}}$$

$$= \frac{(\gamma_1 + \alpha\gamma_2)}{(1-\alpha)\cdot(\gamma_1+\gamma_2)} \cdot E\{\tau_1\} + \frac{E\{\tau_2\}}{(1-\alpha)}$$

⇒ I can express this via the GENERAL FORMULA:



$$\Lambda = \gamma_1 + \gamma_2$$

(Back to the same setting: $\lambda_1 + \gamma_2 = \lambda_2$)

$$\begin{cases} \gamma_1 + \gamma_2 = \lambda_2 (1-\alpha) \\ \gamma_1 + \alpha \lambda_2 = \lambda_1 \end{cases}$$

$$\Rightarrow \lambda_2 = \frac{\gamma_1 + \gamma_2}{1-\alpha}$$

$$\lambda_1 = \gamma_1 + \alpha \frac{\gamma_1 + \gamma_2}{1-\alpha} = \cancel{\frac{\gamma_1 - \alpha\gamma_1 + \alpha\gamma_1 + \alpha\gamma_2}{(1-\alpha)}}$$

$$\lambda_1 = \frac{\gamma_1 + \alpha\gamma_2}{1-\alpha}$$

$$\Rightarrow E\{T\} = \frac{\gamma_1 + \alpha\gamma_2}{(1-\alpha)(\gamma_1+\gamma_2)} \cdot E\{T_1\} + \frac{\cancel{(\gamma_1+\gamma_2)}}{(1-\alpha)\cancel{(\gamma_1+\gamma_2)}} \cdot E\{T_2\}$$

(No need to know probabilities!)

DIFFERENCE WITH REALITY:

Lengths not exponential (random)

↳ Same length of packet over network.

& same service time over all routers

* By mathematical models, different service time at different routers